Mirror symmetry and VOA

Lecture I

Loop-coherent sheaves

R - commutative algebra (local coord. ring)

R-loop-module is a vector space $V = \bigoplus_{\ell \geq 0} V_\ell$

L_0 - grading operator $r[\ell] : V \to V$

a) $1[k] = \delta_k^0$

b) $r[\ell]$ commute with each other

c) $r[\ell] V_k \subset V_{k-\ell}$

$$\left(\sum_k r_1[k] z^{-k}\right)\left(\sum_\ell r_2[\ell] z^{-\ell}\right) = \sum_k (r_1 r_2)[k] z^{-k}$$

$$(r_1 r_2)[0] \neq r_1[0] r_2[0]$$

Proposition S - multiplicative system in R

V_S - localization w.r.t S loop, generated by $s[0]$

Then V_S has a natural structure of R_S-loop module

$\rho : V \to V_S$ - universal morphism

$\forall$ $\rho_1 : V \to V_1$ wich is compatible with $R \to R_S$

$\rho_1 = \rho_2 \circ \rho$

__Def.__ A sheaf V of vector spaces over $\mathbb{C}$ is called quasi-loop-coherent if $\forall$ affine subset $\mathrm{Spec}(R) \subset X$ sections $\Gamma(\mathrm{Spec}(R), V)$ form an R-loop-module and restriction maps are exactly the localization maps.

__Proposition__ $\forall$ R-loop module, consider filtration

$$F^l V = \sum_{i, s_1 \ldots s_i, k_1 \ldots k_i} \prod_i s_i[k_i] V_{\leq l}$$

$$F^0 V \subseteq F^1 V \subseteq \ldots, \quad F^{l+1} V / F^l V$$ has a nat. structure of R-module. (commutes with localizations) __Proof.__ $(s_1 s_2)[0] - s_1[0] s_2[0] : F^{l+1} V \to F^l V$

__Def.__ A quasi-loop coherent sheaf is called loop-coherent or "__loco__" if quasicoherent sheaves $F^{l+1} V \cap V_k / F^l V \cap V_k$ are coherent $\forall$ k, l.

__Terminology__ "loco" and "quasi loco"

__Proposition__ $\forall$ affine variety X and quasi-loco sheaf V on it, cohomology spaces $H^i(X, V)$ are zero for $i \geq 1$. $\forall$ projective variety X all cohomology spaces are fin. dim for each eig. of L_0.

__Proof.__ Choose a specific eigenvalue of k of L_0 and then do induction on l in $F^l V \cap V_k$.

Sheaves of VOA

Def. R - comm. algebra over $\mathbb{C}$.
A graded vertex algebra V is called vertex R-algebra
if R is mapped to $L_0 = 0$ comp of V

$$Y(r_1, z) Y(r_2, z) = Y(r_1 r_2, z)$$

L_0 has only nonneg. eigenvalues

Proposition S - multiplicative system, V - vertex R-algebra
$\Rightarrow V_S$ has a natural structure of vertex R_S-algebra

Proof. Borisov

Def. A (quasi-)loco sheaf V of vector spaces over $\mathbb{C}$
is a (quasi-)loco sheaf of vertex algebras if for $\forall \operatorname{Spec}(R) \subset X$
$\Gamma(\operatorname{Spec}(R), V)$ form a vertex R-algebra and restriction
maps are loc. maps from above.

Proposition Cohomology of quasi-loco sheaf of vertex
algebras V has a natural structure of VA
If the structure of conf. algebra is compatible
with localization maps $\Rightarrow H^*(V)$ has a nat. conf. structure

Proof. Use Borcherds definition.

Topological VOA

$a \in V$ s.t. $a_{(0)}^2 = 0$ $L(z) = [Q, b(z)]$

Proposition Cohomology of V w.r.t. $a_{(0)}$ has a structure of VOA.

$$[a_0, Y(b,z)]_\pm = Y(a_{(0)}b, z) \Rightarrow$$

$\Rightarrow$ If b is annih. by $a_0 \Rightarrow Y(b,z)$ commutes with a_0 and conserves kernel and the image of $\underline{\underline{a_0}}$

Chiral de Rham complex as a sheaf of VOA (X smooth variety)

Loco sheaf $MSV(X)$ in local coordinates:

2dim X ferm. fields $\underset{\substack{\uparrow \\ \text{conf. dim} \\ 0}}{\varphi^i(z)}, \underset{\substack{\uparrow \\ \text{conf. dim} \\ 1}}{\psi_i(z)}$

$\{a_i(z), b^i(z)\}$

2dim X bosonic fields

$\{a_i[k], b^j[l]\}_- = \delta_i^j \delta^0_{k+l}$

$\{\psi_i[k], \psi^j[l]\}_+ = \delta_i^j \delta^0_{k+l}$

$b^i(z) = \sum_k b^i[k] z^{-k}$

$a_i(z) = \sum_k a_i[k] z^{-k-1}$

$\psi_i(z) = \sum_k \psi_i[k] z^{-k-1}$

$\varphi^i(z) = \sum_k \varphi^i[k] z^{-k}$

F

$|0\rangle$ - Fock space

$b_n, \psi_m |0\rangle = 0$ $n > 0, m > 0$

$F \otimes \mathbb{C}[b^i[0]]$ plugged instead of $\{x^i\}$

$\tilde{x}^i = g^i(x)$, $x^i = f^i(\tilde{x}^j)$

↳ $\tilde{b}^i(z) = g^i(b(z))$

$$\tilde{\varphi}^i(z) = g^i_j(b(z))\varphi^j(z)$$

$$\tilde{a}_i(z) = :a_j(z) f^j_i(b(z): + :\psi_k(z) f^k_{i,l} g^l_r(b(z))\varphi^r(z):$$

$$\tilde{\psi}_i(z) = \psi_j f_i^j(b(z))$$

$$g^i_j = \frac{\partial g^i}{\partial x^j} \qquad f^i_j = \left(\frac{\partial f^i}{\partial \tilde{x}^j}\right)\circ g \; , \; f^i_{j,k} = \frac{\partial^2 f^i}{\partial \tilde{x}^j \partial \tilde{x}^k}\circ g$$

$$L(z) = :\partial_z b^i a_i(z): + :\partial_z \varphi^i(z)\psi_i(z):$$

$$J(z) = :\varphi^i(z)\psi_i(z):$$

$$G(z) = \partial_z b^i(z)\psi_i(z)$$

$$Q(z) = \partial_z a_i(z)\varphi^i(z)$$

$L(z)$ - invariant under coord. change. If X - CY all other fields are well-defined

$L[0] = 0$ ≈ isom. to de Rham complex where grading is given by $J[0]$ and diff $Q[0]$

Thus giving a sheaf of TVOA

<u>Def.</u> Let X be a smooth alg. variety over $\mathbb{C}$. We define A-model TVOA over X to be $H^*(MSV(X))$. This algebra also possesses conf. structure $L(z)$ and TVOA str. if X is CY.

Def. If X - CY. Define B-model TUOA as follows:

As a vector space it coincides with the A-model TUOA of X is related by mirror involution:

$$Q_B = G_A \,, \quad G_B = Q_A \,, \quad J_B = -J_A \,, \quad L_B = L_A - \partial J_A$$

B-model - ill-defined if X is not CY