Lecture II

Vertex algebras of line bundles and zeros of their sections

L - line bundle over P (smooth variety)

Alowed local change of coordinates: $\tilde{x}^1 = x^1 h(x^2, \dots, x^{\dim L})$.

$\tilde{x}^i = x^i$, $i \geq 2$

and $x^i = g(x^e)$ $i, e \geq 2$

<u>Proposition</u> $b^1(z)\varphi_1(z)$ depends only on the line bundle structure of L.

$\tilde{b}^1 = h(b^2, \dots, b^{\dim L})$, $\tilde{\varphi}_1 = \varphi_1 / h(b^2, \dots, b^{\dim L})$

<u>Lemma</u> Line bundle L has trivial canonical class iff L is the canonical line bundle on P

$MSV(L)$ is a local sheaf $\Rightarrow$ $H^*(MSV(L)) \cong H^*(\underset{\substack{\text{push-forward}\\ \text{to } P}}{\pi_*} MSV(L))$

Suppose we have μ-section of L^{-1}.
This gives a function on L, which is linear on fibers. Assume that zeros of μ form a reduced non-sing. divisor X on P

We will describe $MSV(X)$ in terms of $\pi_* MSV(L)$

Lemma $\mu(z)$, $D\mu(z)$ defined as

$$\mu(z) = \mu(b^1, \dots, b^{\dim L})(z) \quad D\mu(z) = \sum_i \varphi^i(z) \frac{\partial \mu}{\partial b^i}(z)$$

are indep. of the choice of coordinates and therefore, globally defined.

In particular $BRST_\mu = \oint D\mu(z)dz$ is glob. defined

Theorem X - smooth hypersurface in a smooth variety P defined above by μ. Then $MSV(X)$ is $\cong$ BRST cohomology of $\pi_* MSV(L)$ w.r.t. operator $BRST_\mu$.

Denote BRST cohomology of $\pi_* MSV(L)$ by $\widehat{MSV}(X)$

It is enough to have isomorphism locally $\forall p \in P$

1) $p \notin X$

U - neighb. of p: $p \in U \subset P$ $\quad \underbrace{(x^1, x^2, \dots, x^{\dim L})}_{\text{spec. coord. system}}$ on $\pi^{-1}U$

so that $\mu = x^1 \Rightarrow$

$\Rightarrow BRST_\mu = \oint \varphi^1(z)dz = \varphi^1[1]$

Cohomology is zero $\forall$ small $U \Rightarrow \widehat{MSV}(X)$ is supported on X, which is true for $MSV(X)$

2) $p \in X$ On a suff. small neighb. $U \subset P$ system of coord: $(x^1, \dots, x^{\dim L})$ on $\pi^{-1}U$

$\mu = x^1 x^2 \quad \underbrace{x^3, \dots, x^{\dim L}}$ form system of coord. on $X \cap U$

Then $BRST_\mu = \oint (b^1 \varphi^2 + b^2 \varphi^1)(z)\, dz =$

$$= \sum_{k \in \mathbb{Z}} \left(b^1[k] \varphi^2[-k+1] + b^2[k] \varphi^1[-k+1] \right)$$

Fock space $\quad P(U, \pi_x\, HSV(\lambda)) = \mathrm{Fock}_{1,2} \otimes \mathrm{Fock}_{\geq 3}$

generated $\{a_i, b^i, \varphi^i, \psi_i\}_{i=1,2}$ $\qquad i \geq 3$

Cohomology $\cong H_{BRST_\mu}(\mathrm{Fock}_{1,2}) \otimes \mathrm{Fock}_{\geq 3}$

<u>Claim:</u> $H_{BRST_\mu}(\mathrm{Fock}_{1,2})$ 1-dim and generated by $|0\rangle$

$\mathrm{Fock}_{1,2}$ is a restr. tensor product of

- $\bigoplus_{\ell \geq 0} \mathbb{C}(a^1[-k])^\ell \otimes \bigoplus_{\ell \geq 0} \mathbb{C}(a^1[-k])^\ell \varphi^2[-k+1] \quad \forall k > 0$
- $\bigoplus_{\ell \geq 0} \mathbb{C}(a^2[-k])^\ell \otimes \bigoplus_{\ell \geq 0} \mathbb{C}(a^2[-k])^\ell \varphi^1[-k+1] \quad \forall k > 0$
- $\bigoplus_{\ell \geq 0} \mathbb{C}(b^1[-k])^\ell \otimes \bigoplus_{\ell \geq 0} \mathbb{C}(b^1[-k])^\ell \psi^2[-k-1] \quad \forall k \geq 0$
- $\bigoplus_{\ell \geq 0} \mathbb{C}(b^2[-k])^\ell \otimes \bigoplus_{\ell \geq 0} \mathbb{C}(b^2[-k])^\ell \psi^1[-k-1] \quad \forall k > 0$
- $\bigoplus_{\ell \geq 0} R(b^2[-k])^\ell \otimes \bigoplus_{\ell \geq 0} R(b^2[-k])^\ell \psi^1[-k-1] \quad \forall k > 0$

R - ring of functions on a disc

U is a product of $|x_2| \leq C$ and $U_{x^3 \ldots x^{\dim L}}$

$F_{1,2}$ is graded by $\lambda_{1,2}[0]$

$BRST_\mu$ shifts grading by -1

<u>Ex.</u> Finish the proof of claim

We showed that for a given choice of coord. on $U \cap U$ there is an isom. between sections of $\widehat{MSV}(X)$ and $MSV(X)$

One has to show that these isom. could be glued together.

One has to show that the constructed isomorphism commutes with change of coordinates pres. our setup

$$\hat{x}^1 = x^1 \cdot h(x^2, \dots, x^{\dim L}), \quad \tilde{x}^2 = x^1 / h(x^2, \dots, x^{\dim L})$$

$$\hat{x}^i = f^i(x^3, \dots, x^{\dim L}) + x^2 g^i(x^2, \dots, x^{\dim L}), \quad i \geq 3$$

$h = 1$, $g^i = 0$ – splitting is unaffected

so, only have to show that for f^i-nontrivial

but in this case $\hat{a}_i$, $\hat{b}^i(z) \dots$ $i \geq 3$ act on the cohomology as $a_i, b^i \dots$ $i \geq 3$ since the diff. lies in the image.

<u>Prop.</u> $\forall$ affine subset $U \subset P$, $BRST_\mu$ coh. space of $\Gamma(U, \pi_* MSV(L))$ is isom. to $\Gamma(U, MSV(X))$

Consider the case when L has non-deg. top form

$\Rightarrow$ L-canonical line bundle, section μ of L^{-1} produces CY divisor X on P. Let's compute $G(z)$ and $Q(z)$ on $MSV(X)$ in terms of global fields on $\pi_* MSV(L)$

<u>Prop.</u> L- canonical bundle on $X \Rightarrow G_X(z)$ is the image of $G_L(z) - (b_1(z)\,\varphi_1(z))'$, $Q_X(z)$ is the image of $Q_L(z)$

<u>Proof</u> Local computation

$$Q_X(z) - Q_L(z) = -a_1(z)\varphi^1(z) - a_2(z)\varphi^2(z)$$

$$G_X(z) - G_L(z) + (b^1(z)\,\psi_1(z))' = b^2(z)'\psi_2 - b^1(z)\psi_1(z)'$$

We need to show that the RHS are given by $[Q_{BRST}, A(z)]$

1) $A(z) = -a_1(z)\,a_2(z)$

2) $A(z) = -\psi_1'(z)\,\psi_2(z)$