Lecture III Logarithmic coordinates

M - free abelian group of rank $\dim M$, $N = Hom(M, \mathbb{Z})$

$(M \oplus N) \otimes \mathbb{C}$ is of $\dim = 2 \dim M$, has standard bil. form

We construct $2\dim M$ bosonic and $2\dim M$ fermionic fields:

$$m \cdot B(z) = \sum_{k \in \mathbb{Z}} m \cdot B[k] z^{-k-1}, \quad n \cdot A(z) = \sum_{k \in \mathbb{Z}} n \cdot A[k] z^{-k-1}$$

$$m \cdot \Phi(z) = \sum_{k \in \mathbb{Z}} m \cdot \Phi[k] z^{-k}, \quad n \cdot \Psi(z) = \sum_{k \in \mathbb{Z}} n \cdot \Psi[k] z^{-k-1}$$

$$[m \cdot B[k], n \cdot A[l]]_- = (m \cdot n) k \, \delta_{k+l,0} \, id$$

$$[m \cdot \Phi[k], n \cdot \Psi[l]]_+ = (m \cdot n) \, \delta^0_{k+l} \, id$$

$\mathrm{Fock}_{M \oplus N}$: Vac. vectors $|m,n\rangle$, s.t.

$$A[0]\,|m,n\rangle = m\,|m,n\rangle$$

$$B[0]\,|m,n\rangle = n\,|m,n\rangle$$

annih. by all positive modes and $\Phi[0]$

Vertex operators:

$$V_{m,n}(z) = :e^{\int (m \cdot B(z) + n \cdot A(z))}:$$

$$V_{m,n}(z) \left(\prod A \prod B \prod \Phi \prod \Psi \, |m_1, n_1\rangle \right) =$$

$$= C(m,n,m_1,n_1) \, z^{m \cdot n_1 + n \cdot m_1} \prod_{n<0} e^{-(m \cdot B[n] + n A[n]) \frac{z^n}{n}} \prod_{n>0} e^{-(m \cdot B[n] + n A[n]) \frac{z^n}{n}}$$

Cocycle: $C(m,n,m_1,n_1) = (-1)^{m\cdot n_1}$ – cocycle to make all of them bosonic.

We will suppress it in the following.

$$V_{m,n}(z)\, V_{m_1,n_1}(\omega) = \frac{:V_{m,n}(z)\, V_{m_1,n_1}(\omega):}{(z-\omega)^{m\cdot n_1 + n\cdot m_1}}$$

Conformal structure

$\mathrm{Fock}_{M\oplus N}$ $\quad L_{M\oplus N}(z) = \,:B(z)\cdot A(z): + :\partial_z \Phi(z)\cdot \Psi(z):$

$|m,n\rangle$ has grading $\underline{m\cdot n}$

Bosonization formulas: (dim 1)

$$b(z) = e^{\int B(z)}, \quad \varphi(z) = \Phi(z)\, e^{\int B(z)}, \quad \psi(z) = \Psi(z)\, e^{-\int B(z)}$$

$$a(z) = \,:A(z)\, e^{-\int B(z)}: + :\Phi(z)\Psi(z)\, e^{-\int B(z)}:$$

Proposition $\quad a(z)\, b(\omega) = \dfrac{1}{z-\omega} + \text{reg}$

$$\varphi(z)\, \psi(\omega) = \frac{1}{z-\omega} + \text{reg}.$$

Proposition $\quad L(z),\ J(z),\ Q(z),\ G(z)$

$$Q(z) = A(z)\,\Phi(z) - \partial_z \Phi(z), \quad G(z) = B(z)\,\Psi(z)$$

$$J(z) = \,:\Phi(z)\Psi(z): + B(z)$$

$$L(z) = \,:B(z)A(z): + :\partial_z \Phi(z)\, \Psi(z):$$

Define $\mathrm{Fock}_{M\oplus N\geq 0}$ – eig. of $B[0]$ is nonnegative.

$|m,n\rangle \quad n \geq 0$

Theorem Vertex algebra of a, b, φ, ψ is isomorphic to $\mathrm{Fock}_{M\oplus N\geq 0}$ w.r.t

$$BRST_g = \oint BRST_g(z) = \int g\,\Psi(z)\, e^{\int A(z)}\, dz$$

$g \in \mathbb{C}$

Proof. 1) Show $[a(z), BRST_g] = 0$ as well as all other fields

2) $[R(z), BRST_g] = id$, where $R(z) = \bar{\Phi}(z)\, e^{-\int A(z)}$

$\Rightarrow [R[0], BRST_g] = id$ - homotopy operator which kills all cohomology in $B[0] \neq 0$ since $BRST_g(z)$ shifts them by 1.

3) by induction on eig. of $h[0]$ that all the elements of the kernel are generated by a, b, φ, ψ

Extending to any dimension:

Consider primitive cone K^* in lattice N

Choose basis $n_1, \dots, n_{\dim N}$

dual $m_i \dots m_{\dim M}$

$\mathrm{Fock}_{M\oplus K^*}$

$$b^i(z) = e^{\int m_i B}, \quad \varphi^i(z) = (m_i \cdot \bar{\Phi}(z))\, e^{\int m_i B(z)}$$

$$\psi_i = (n_i \cdot \Psi(z))\, e^{-\int m_i B(z)}$$

$$a_i(z) = :n_i A(z)\, e^{-\int m_i B(z)}: + :m_i \cdot \bar{\Phi}(z)\, n_i \Psi(z)\, e^{-\int m_i B(z)}$$

$i = 1, \dots, \dim M$

Theorem Vertex algebra $a_i, b^i, \varphi^i, \psi_i$ $\cong$

$$H_{BRST_g} = \oint \sum_i g_i (u_i \psi(z)) e^{\int u_i A(z)} dz$$

$g_1 \ldots g_{dim}$ - arbitrary in $\mathbb{C}$

$$Q = A \cdot \Phi - \underline{deg} \cdot \partial_z \Phi \qquad G = B \cdot \Psi$$

$$J = :\Phi \cdot \Psi: + \underline{deg} \cdot B$$

$$L(q) = :B \cdot A: + :\partial_z \Phi \cdot \Psi:$$

deg - element of M, which equals 1 on all gen. of K^*