Lecture II Chiral vertex operators: properties.

$$\Phi^{j,\beta}_{ik}(z) : \mathcal{H}_k \to \mathcal{H}_i\{z\} \qquad \beta \in \mathcal{H}_j$$

$$\langle \alpha | \phi^{j}_{ik}(z) | \beta \rangle \in z^{-(h_j+h_k-h_i)}\,\mathbb{C}[z,z^{-1}]$$

Something similar to VOA:

$$\Phi^{j,\cdot}_{ik}(z) : \mathcal{H}_j \otimes \mathcal{H}_k \to \mathcal{H}_i\, z^{-(h_j+h_k-h_i)}[z,z^{-1}]$$

Redefine conformal block

$$\mathcal{F}^{ijkl}_{p}(z_2,z_3) = \langle i | \Phi^{j,\alpha}_{ip}(z_2)\, \Phi^{k,\beta}_{pl}(z_1) | l \rangle$$

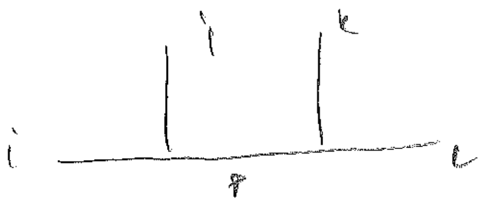
$\sim$ 

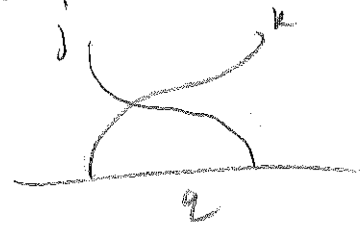
constants

$$\langle \phi^i | \phi^j(z_2)\, \phi^k(z_3) | \phi^l \rangle = \sum_p d_p |\mathcal{F}_p|^2$$

correlation function

$$\Phi^{j,\alpha}_{ip}(z_1)\, \phi^{k,\beta}_{pl}(z_2) = \sum_q B_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \Phi^{k,\alpha}_{iq}(z_2)\, \Phi^{j,\beta}_{ql}(z_1)$$



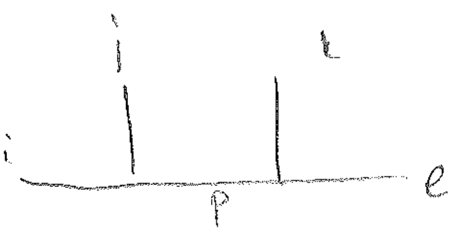
$= \sum_q B_{pq}$ 

choose the cut: $z_1 - z_2 \in \mathbb{R}^+$

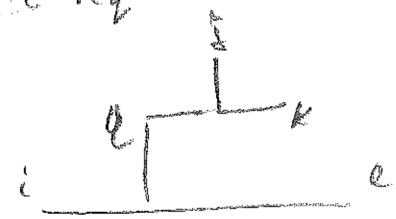
Fusion operation:

take OPE of $\phi^i(z_1)\,\phi^k(z_2)$

$$\Phi^{j,\alpha}_{ip}(z_1)\,\Phi^{k,\beta}_{pe}(z_2)=\sum_q F_{pq}\begin{bmatrix} i & k\\ i & e\end{bmatrix}\sum_{\gamma\in H_q}\Phi^{q,\gamma}_{ie}(z_2)\langle\gamma|\Phi^{j,\alpha}_{qk}(z_{12})|k\beta\rangle$$



$$=\sum_q F_{pq}\begin{bmatrix} i & k\\ i & e\end{bmatrix}$$



$$[\phi_i]\cdot[\phi_j]=N_{ij}^{k}\,[\phi_k]$$

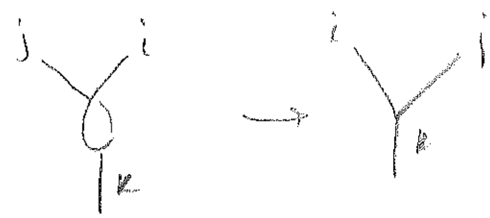
$\uparrow$ nonnegative integers

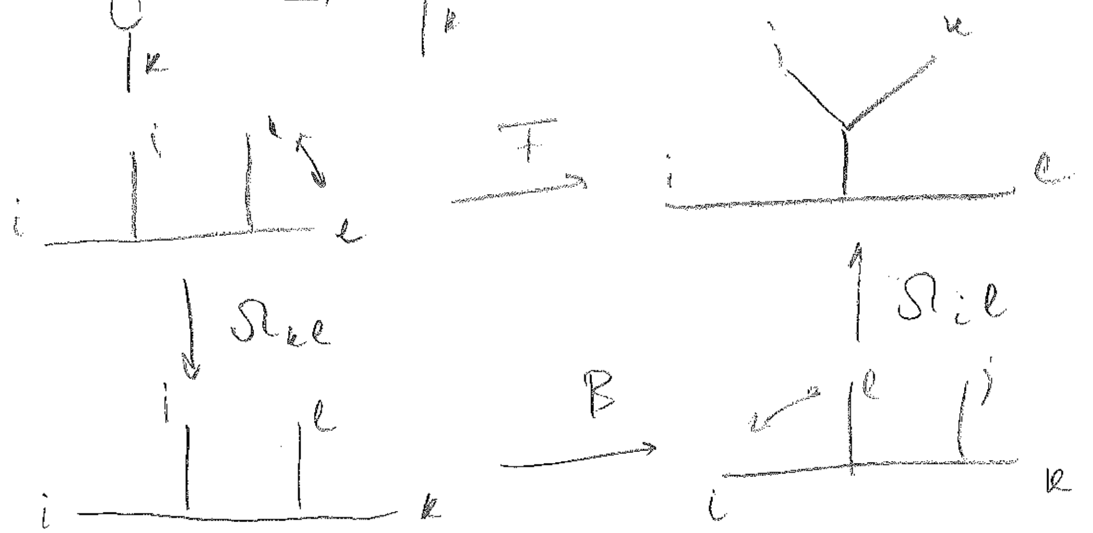
$[N_i, N_j]=0$ – from associativity

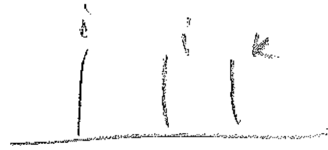
<u>Exercise:</u> $B\begin{bmatrix} i & i\\ 0 & k\end{bmatrix}$ is a map: $V^i_{ik}\to V^i_{ki}$

$e^{\pi i(h_i+h_k-h_i)}\,\Omega^i_{ik}$

$(\Omega^i_{ik})^2=e^{\ldots}$





B - hexagon identity 

F – pentagon indentity

Modular tensor category (return back later)

Degenerate conformal families and Null-vectors

$\{ L_{-n_1} \dots L_{-n_N} |h\rangle \}$ span.

$$\| L_{-1}|h\rangle \|^2 = \langle h | L_1 L_{-1} |h\rangle = 2h > 0$$

$$\| L_{-N} |h\rangle \|^2 > 0 \quad \Rightarrow c > 0$$

Null vectors?

$$L_n |\chi\rangle = 0 \quad \forall n > 0$$

$$L_0 |\chi\rangle = (h+N)|\chi\rangle$$

Let's find one on level $N=2$

$$(a L_{-2} + b L_{-1}^2)|h\rangle$$

L_1, L_2 - annihilate $\rightarrow$ produces all constraints

$$|\chi_h\rangle = \left(L_{-2} - \frac{3}{2(h+1)} L_{-1}^2 \right)|h\rangle$$

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$$

Ex. Ward identity: $\langle T(w)\, \phi_h(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle =$

$= \dots$

$\langle \chi_h(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle =$

$$= \left(\frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N} \frac{h_i}{(z-z_i)^2} - \sum_{i=1}^{N} \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_h(z) \phi_1 \dots \phi_N \rangle$$

4-point function conformal block is given by hypergeometric diff. equation, whose solutions will be given by contour integrals.

Kac determinant gives information of all null vectors.

Param: $c = 1 - 24\alpha_0^2$ $\qquad \alpha_0 = \alpha_+ + \alpha_-$, $\alpha_+ \alpha_- = -1$

$$h(n,m) = -\frac{1}{4}\alpha_0^2 + \frac{1}{4}(n\alpha_+ + m\alpha_-)^2$$

$n, m \in \mathbb{Z}$

$V_{n,m} = V(c, h_{n,m})$ has null vector at level nm

Let $h = h(1,2)$ or $h(2,1)$

$$\phi_h(z)\,\phi_{h_1}(z_1) = \text{const}\,(z - z_1)^k \left(\phi_{h'}(z_1) + b\,(z - z_1)\,\phi^{(-1)}_{h'}(z_1) + \dots\right.$$

Then differential equation

$$\frac{3}{2(2h+1)}\,k(k-1) - h_1 + k = 0$$

scale invariance: $k = h' - h_1 - h$

$$h_1 = -\frac{1}{4}\alpha_0^2 + \frac{1}{4}\alpha^2 \qquad h' = -\frac{1}{4}\alpha_0^2 + \alpha'^2$$

we find $h = h(1,2)$ $\qquad \alpha' = \alpha \pm \alpha_-$

$h = h(2,1)$ $\qquad \alpha' = \alpha \pm \alpha_+$

Fusion rules: $\phi_{(1,2)}\,\phi_{(\alpha)} = [\phi_{\alpha - \alpha_-}] + [\phi_{\alpha + \alpha_-}]$

$\phi_{(2,1)}\,\phi_{(\alpha)} = [\phi_{\alpha - \alpha_+}] + [\phi_{\alpha + \alpha_+}]$

$$\phi_{(1,2)}\,\phi_{(1,2)} = [\phi_{1,1}] + [\phi_{1,3}]$$

$$\phi_{(2,1)}\,\phi_{(2,1)} = [\phi_{1,1}] + [\phi_{3,1}]$$

$$\phi_{(1,2)}\,\phi_{(1,m)} = [\phi_{(1,m-1)}] + [\phi_{(1,m+1)}]$$

$$\phi_{(2,1)}\,\phi_{(m,1)} = [\phi_{(m-1,1)}] + [\phi_{(m+1,1)}]$$

Only $\phi_{(n,m)}$ $n,m>0$ appear

Truncation:

$$\phi_{(1,2)}\,\phi_{(2,1)} = c_1[\phi_{(2,0)}] + c_2[\phi_{(2,2)}] \Rightarrow$$

$$\phi_{(1,2)}\,\phi_{(2,1)} = c_1'[\phi_{(0,2)}] + c_2'[\phi_{(2,2)}]$$

$$\Rightarrow \phi_{(1,2)}\,\phi_{(2,1)} = [\phi_{(2,2)}]$$

$$\phi_{(n_1,m_1)}\,\phi_{(n_2,m_2)} = \sum_{k=(n_1-n_2)+1}^{n_1+n_2-1}\ \sum_{l=(m_1-m_2)+1}^{m_2+m_1-1} [\phi_{(k,l)}]$$