Lecture I (New)

What this course is about?

QFT:

$$\frac{1}{Z}\int \underbrace{f_1(\varphi)\ldots f_n(\varphi)}_{\text{observables}}\, e^{-\overbrace{S[\varphi]}^{\text{action}}}\, \underbrace{[d\varphi]}_{\text{fields}} =$$

partition function

$$=: \langle f_1 \ldots f_n \rangle \leftarrow \text{correlation function}$$

Computing such integrals is a primary target of QFT.

Topological Field Theories

→ exact results, using fixed point formulas and other techniques (Correlation functions give various invariants: Donaldson, GW, Jones polynomial, etc

Conformal Field Theories in 2D: (BPZ)

$$A_i(x_i)\, A_j(x_j) = C_{ij}^{\{k\}}(x_i, x_j)\, A_{\{k\}}(x_j)$$

operator product expansion

Infinite-dimensional symmetry (Virasoro algebra) lead to constraints on $C_{ij}^{\{k\}}(x_i, x_j)$ expressed via diff. equations

Structure of the course

1) Geometric (Segal) and Constructive (BPZ) approach

2) Vertex algebras (simplest CFTs)

3) Rational conformal field theories (minimal models, WZW). Relationship to Quantum Groups.

4) Advanced topics:
1) Chiral de Rham complex. Applications to Mirror symmetry
2) SLE approach to CFT
3) Integrable models and CFT.
4) Applications to Geometric Langlands

Geometric approach, applied to QFT in general

$D+1$-dimensional QFT, among other things is a functor $\Phi : \underline{Man(D)} \longrightarrow \underline{Vect}$

- "manifold" - loose notion, correct is $*$-manifold where $*$ is the extra structure

- $M \sim M'$ if there is a diff., preserving $*$-structure

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Field Theories: topological, spin, conformal $*$: orientation, spin structure, complex structure

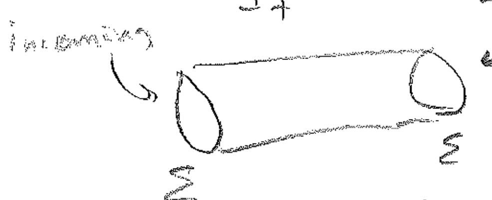
<u>2 ideas:</u>

i) Σ - closed D-dim manifold $\to$ Hilbert space $\mathcal{H}_\Sigma$

Physically: space of quantum states obtained by quantizing theory on $\Sigma \times \mathbb{R}$ spacetime.

ii) "Evolution operator" $\quad \Phi_{t+t'} = \Phi_t \circ \Phi_{t'}$

$$\Phi_t : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$$



$\partial M = (\underset{\text{outgoing}}{\Sigma}) \cup (\underset{\text{incoming}}{-\Sigma})$

In general to any M with two boundaries Σ and Σ'

$$\Phi_M : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

Φ_M from physics perspective: $\{\phi_i\}$ - local set of fund fields

$$\Psi_{out}(\phi') = \int D\phi \, K_M(\phi', \phi) \Psi_{in}(\phi) \qquad K_M(\phi', \phi) = \int_{\substack{\varphi|_\Sigma = \phi \\ \varphi|_{\Sigma'} = \phi'}} D\varphi \, e^{-S(\varphi)}$$

Gluing $\quad \Phi_{M' \circ M} = \Phi_{M'} \circ \Phi_M$

$$K_{M' \circ M}(\phi'', \phi) = \int D\phi' \, K_{M'}(\phi'', \phi') K_M(\phi', \phi)$$

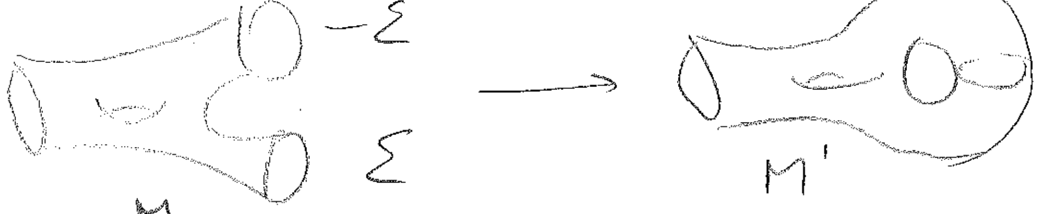
<u>Axioms:</u>

1) In degenerate case Σ consists of $\emptyset \Rightarrow \mathcal{H}_\emptyset = \mathbb{C}$

2) If Σ is a disjoint union of several manifolds

$$\mathcal{H}_{\Sigma \amalg \Sigma'} = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\Sigma'}$$

3) The "collapsing" axiom: if M has two boundary $\angle \Sigma$ components $-\Sigma$ and Σ, labeled as outgoing and incoming, one can glue the boundaries



$$\Phi_{Tr_\Sigma M'} = Tr_{V_\Sigma} \Phi_M = \sum \Phi_M (v_i, v^i)$$ v_i, v^i conj. bases in $\mathcal{H}_\Sigma, \mathcal{H}_\Sigma^*$

4) $\mathcal{H}_{-\Sigma} \cong \mathcal{H}_\Sigma^*$ or if $\mathcal{H}_\Sigma$ carries Herm. structure

$$\mathcal{H}_{-\Sigma} \cong \overline{\mathcal{H}_\Sigma}$$

5) A natural action of perm. group, if several boundary components are isomorphic

Remarks: 1) Closed $D+1$-manifold M $\Phi_M : \mathbb{C} \to \mathbb{C}$ is $Z(M)$ - partition function

2) If M has only one single outgoing boundary $\Sigma \Rightarrow \Phi_M$ is a map $\mathbb{C} \to \mathcal{H}_\Sigma$ $|M\rangle \in \mathcal{H}_\Sigma$

$$\Psi_M(\phi_\Sigma) = \int_{\phi|_\Sigma = \phi_\Sigma} D\phi\, e^{-S(\phi)}$$

Sueing M_1, M_2 into M

$$Z(M) = \langle M_1 | M_2 \rangle = \int D\phi\, \Psi_{M_1}(\phi) \Psi_{M_2}(\phi)$$

Topological FT

One can glue M_1, M_2 via surgery (applying $\mathrm{Diff}(\Sigma)$)

In topological theory $|M\rangle \in \mathcal{H}_\Sigma$ is inv under $\mathrm{Diff}_0(\Sigma)$ since it can always be deformed over a collar $\Sigma \times I$ $\Rightarrow$ only Γ_Σ (mapping class group) act

$$1 \to \mathrm{Diff}_0(\Sigma) \to \mathrm{Diff}(\Sigma) \to \Gamma_\Sigma \to 1$$

$$Z(M') = \langle M_2 | U(\gamma) | M_1 \rangle \qquad \gamma \in \Gamma_\Sigma$$

Also $\dim \mathcal{H}_\Sigma = \mathrm{Tr}_{\mathcal{H}_\Sigma} 1 = Z(\Sigma \times S^1)$ if $\mathcal{H}_\Sigma$ f.d.

2d TFT:

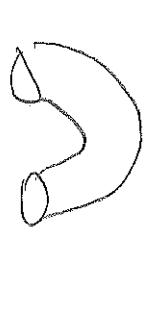
Only one closed manifold: S^1 $\quad \mathcal{H} = \mathcal{H}_{S^1}$ or $\mathcal{H}^* = \mathcal{H}_{S^1}$

Φ_M??? They can be defined by considering the sphere with 1, 2, 3 holes

1)  $\mathbb{C} \to \mathcal{H}$ $\Rightarrow$ we call it vacuum vector $\mathbb{1}$.

 $\mathcal{H} \to \mathbb{C}$ $\Rightarrow$ $\langle\ \rangle_0 : \mathcal{H} \to \mathbb{C}$

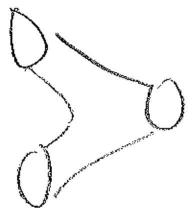
$$Z(S^2) = \langle \mathbb{1} \rangle_0$$

2)  $\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ $\qquad \{\phi_i\}$ - basis

$$\eta_{ij} = \eta(\phi_i, \phi_j)$$

Ex. Show that it is not degenerate.

 η^{ij} - inverse matrix

3)  $c : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

$$c : \mathcal{H} \to \mathrm{End}(\mathcal{H})$$

$$\alpha \cdot \beta = c(\alpha, \beta)$$

$$\phi_i \cdot \phi_j = \sum c_{ij}^{k} \phi_k$$

 $c : \mathrm{Sym}^3 V \to \mathbb{C}$

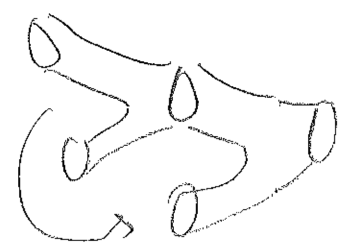
$$c_{ijk} = c_{ij}^{\ell} \eta_{\ell k} \qquad \eta_{ij} = c_{ij0}$$

Picture

$$\eta(\alpha \cdot \beta, \gamma) = \eta(\alpha, \beta \cdot \gamma)$$

Also:

$$(\alpha \cdot \beta) \cdot \gamma = \alpha(\beta \cdot \gamma)$$



$\eta(\alpha, \beta) = \langle \alpha \cdot \beta \rangle_0$

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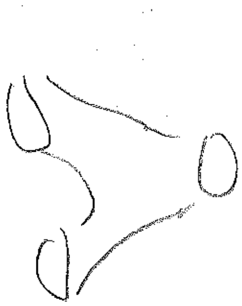
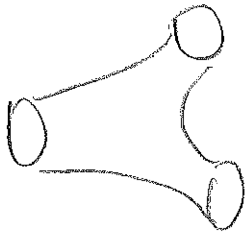
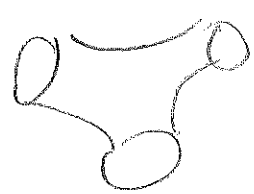
Duality:

ϕ_j ϕ_i   ϕ_k ϕ_ℓ = ϕ_j ϕ_k   ϕ_i ϕ_ℓ

$$\sum_n C_{ij}{}^n C_{nkl} = \sum_n C_{jk}{}^n C_{nil}$$

$(\phi_i)_j{}^k = C_{ij}{}^k$ – symmetric commuting matrices can be simultaneously diagonalized by $S_i{}^j$ that is orthogonal w.r.t. η_{ij}

$$C_{ij}{}^k = \sum_n S_j{}^n \lambda_i^{(n)} (S^{-1})_n{}^k \Rightarrow \lambda_i^{(n)} \lambda_j^{(n)} = \sum_k C_{ij}{}^k \lambda_k^{(n)}$$

$$C_{0j}{}^k = \delta_j{}^k \Rightarrow \lambda_i^{(n)} = \frac{S_i{}^n}{S_0{}^n}$$

$$\delta_j{}^k = \sum_n S_0{}^n \lambda_j^{(n)} (S^{-1})_n{}^k$$

$$S_0^n \lambda_j^{(n)} = S_j^n$$

$$C_{ij}{}^k = \sum_n \frac{S_i{}^n S_j{}^n (S^{-1})_n{}^k}{S_0{}^n}$$ Verlinde formula

Exercises:

1) Calculate $Z(M)$ for genus g → 2g-2 3-punctured spheres in terms of S

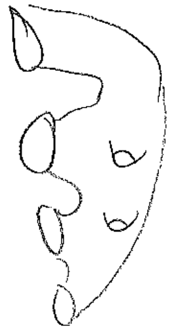
$$\tilde{\phi}_i = (S^{-1})_i{}^j \phi_j$$

$$Z(M) = \sum_k (S_0^k)^{2(1-g)} \qquad \tilde{\phi}_i \times \tilde{\phi}_j = \frac{\delta_{ij}}{S_0^i} \tilde{\phi}_i \qquad Z(S^1 \times S^1) = \dim \mathcal{H}$$

2) Calculate partition functions

$$\Phi_M : |\phi_{i_1}\rangle \otimes \ldots |\phi_{i_n}\rangle \to \mathbb{C}$$

$$\langle \phi_{i_1} \ldots \phi_{i_n} \rangle_M = \sum_k S_{i_1}^k \ldots S_{i_n}^k (S_0^k)^{2(1-g)-n}$$



n-point functions