Lecture II (New)

Complex structures and metrics in 2d

Complex structure on RS: $J^a_b = \sqrt{g}\,\varepsilon_{ac}\,g^{cb}$ $\quad$ integrable $\nabla J = 0$

Invariant under rescaling $g \to e^{\phi} g$, i.e. J defines only the conformal class of metrics

Liouville cocycle: $\mathcal{A} : C^\infty(\Sigma, \mathbb{R}) \times Met(\Sigma) \to \mathbb{R}$

$$\mathcal{A}(\phi_1 + \phi_2, g) = \mathcal{A}(\phi_1, e^{\phi_2} g) + \mathcal{A}(\phi_2, g)$$

Definition:

If g_1, g_2 – two metrics, compatible with complex structure, $g_i = e^{\varphi_i} |dz|^2$

$$S(g_1, g_2) = \frac{1}{48\pi} \int_\Sigma \underbrace{(\varphi_1 - \varphi_2)\,\partial\bar\partial(\varphi_1 + \varphi_2)}_{\text{“} L(g_1, g_2)}$$

Exercise: $S_{Liouville, g}(\sigma) = \frac{1}{12\pi}\left(\frac{1}{2}\|d\sigma\|^2 + \int_\Sigma R(g)\,\sigma\,\mathrm{Vol}\right)$

Prove that $S_{Liouv., g}(\sigma) = - S(g, e^{2\sigma} g)$

Lemma $L(g_1, g_2)$ vanishes at the points, where both g_1, g_2 are flat

Lemma 1) $S(g_1, g_3) = S(g_1, g_2) + S(g_2, g_3)$

2) $L(g_1, g_2) + L(g_2, g_3) + L(g_3, g_1) = d\alpha(g_1, g_2, g_3)$

$$\alpha(g_1, g_2, g_3) = -\frac{1}{6} \sum \epsilon^{ijk} \log\frac{g_i}{g_j} (\partial - \bar\partial) \log \frac{g_j}{g_k}$$

Determinant line:

$|\det|^c_\Sigma$ - oriented 1-dimensional vector space over Σ

$[g_2]/[g_1] = \exp cS(g_2, g_1)$

Segal's axioms of CFT

Setup: Compact Riemann surface Σ (connected or disconnected)

Boundary: $C_i \cong S^1$, $i \in I$, we parametrize S^1 in a real analytic way

$I = I_{in} \sqcup I_{out}$, depending on orientation of S^1's.

Inversion of orientation $z \to z^{-1}$ of S^1

$\Sigma \longrightarrow \hat{\Sigma}$ (compact surface w/o boundary) in a unique way attaching disc $D = \{|z| \leq 1\}$ to C_i $i \in I_{in}$ and

$D' = \{|z| \geq 1\}$ to C_i $i \in I_{out}$

Conversely:

$\hat{\Sigma} \to \Sigma$. Consider $\hat{\Sigma}$ with holomorphically embedded disj. discs D and D' (<u>local parameters</u>). Removing the interiors we obtain Σ.

Metric g is called "flat" at the boundary if near C_i it has the form $|z|^{-2}|dz|^2$ in local holomorphic coordinates extending parametrization of C_i

We will consider only such metrics.

Segal's axioms

i) Hilbert space $\mathcal{H}$, anti-unitary involution $\mathcal{I}$

$$\mathcal{A}_{\Sigma,\gamma} : \bigotimes_{i \in I_{in}} \mathcal{H} \to \bigotimes_{i \in I_{out}} \mathcal{H} \text{ (assume trace-class)}$$

ii) If Σ is a disjoint union of Σ_1 and Σ_2

$$\mathcal{A}_{\Sigma,\gamma} = \mathcal{A}_{\Sigma_1,\gamma} \otimes \mathcal{A}_{\Sigma_2,\gamma}$$

iii) If $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism reducing to identity around the param. boundary, then

$$\mathcal{A}_{\Sigma_2,\gamma} = \mathcal{A}_{\Sigma_1, f^*\gamma}$$

iv) If $\bar{\Sigma}$ denotes RS with conjugate complex structure (and opp. orientation)

$$\mathcal{A}_{\bar{\Sigma},\gamma} = \mathcal{A}^{+}_{\Sigma,\gamma}$$

v) The inversion of boundary param on $\mathcal{A}_{\Sigma,\gamma}$ acts on $\mathcal{A}_{\Sigma,\gamma}$ by $\mathcal{H} \cong \mathcal{H}^*$ induced by $\mathcal{I}$

vi) If Σ' is obtained from Σ by gluing C_{i_0} and C_{i_1} boundary components, then $\mathcal{A}_{\Sigma',\gamma} = \mathrm{tr}_{i_0 i_1} \mathcal{A}_{\Sigma,\gamma}$ where tr_{i_0,i_1} denotes the partial trace in $\mathcal{H}$-factors

vii) $$\mathcal{A}_{\Sigma, e^{\sigma}\gamma} = e^{\frac{c}{96\pi}\left(\|d\sigma\|^2 + 4\int_\Sigma \sigma R\, dv\right)} \mathcal{A}_{\Sigma,\gamma}$$

σ-vanishes on the boundary

Another look: Constructive field theory

(Σ, γ) - compact Riemann surface (w/o boundary)

We associate to (Σ, γ)

$Z_\gamma > 0$ - partition function

primary fields

$\langle \phi_{e_1}(x_1) \dots \phi_{e_n}(x_n) \rangle_\gamma$ - correlation functions

symmetric in the pairs (x_i, e_i)

Defined for non-coincident insertions and assumed smooth.

1) f - Diff.

$$Z_\gamma = Z_{f^*\gamma}$$

$$\langle \phi_{e_1}(f(x_1)) \dots \phi_{e_n}(f(x_n)) \rangle_\gamma =$$

$$= \langle \phi_{e_1}(x_1) \dots \phi_{e_n}(x_n) \rangle_{f^*\gamma}$$

2)
$$Z_{e^\sigma \gamma} = e^{\frac{c}{96\pi}\left(\|d\sigma\|^2 + 4\int_\Sigma \sigma R dv\right)} Z_\gamma$$

$$\langle \phi_{e_1}(x_1) \dots \phi_{e_n}(x_n) \rangle_{e^\sigma \gamma} = \prod_{i=1}^n e^{-\Delta_{e_i} \sigma(x_i)} \langle \phi_{e_1}(x_1) \dots \phi_{e_n}(x_n) \rangle_\gamma$$

c - central charge

3) Under the change of orientation of Σ

$$Z_\gamma \mapsto Z_\gamma$$

$$\langle \phi_{e_1}(x_1) \dots \phi_{e_n}(x_n) \rangle_\gamma \mapsto \langle \phi_{\bar{e}_1}(x_1) \dots \phi_{\bar{e}_n}(x_n) \rangle_\gamma$$

involution, pres. conf. weights

Energy-momentum tensor:

$$\langle T_{\mu_1\nu_1}(y_1)\dots T_{\mu_m\nu_m}(y_m)\,\phi_{e_1}(x_1)\dots\phi_{e_n}(x_n)\rangle_\gamma =$$

$$= Z_\gamma^{-1}\,\frac{(4\pi)^m\,\delta^m}{\delta\gamma^{\mu_1\nu_1}(y_1)\dots\delta\gamma^{\mu_n\nu_n}(y_n)}\,Z_\gamma\langle\phi_{e_1}(x_1)\dots\phi_{e_m}(x_m)\rangle_\gamma$$

Notice: $\gamma^{\mu\nu}\partial_\mu\partial_\nu = \gamma^{-1}$ - inverse metric

Complex coordinates T_{zz}, $T_{\bar z\bar z}$, $T_{z\bar z}=T_{\bar z z}$

By definition corr. functions above are distr. in $y_1,\dots,y_n$

We will show that they are smooth for $\begin{cases} y_i\neq y_j \\ y_i\neq x_j\end{cases}$

Conformal Ward identities

$$4\pi Z_\gamma^{-1}\frac{\delta}{\delta\sigma}\Big|_{\sigma=0} Z_{e^{\sigma}\gamma} = -\gamma^{zz}\langle T_{zz}\rangle_\gamma - 2\gamma^{z\bar z}\langle T_{z\bar z}\rangle_\gamma -$$

$$-\gamma^{\bar z\bar z}\langle T_{\bar z\bar z}\rangle_\gamma = \frac{c}{6}R$$

If $\gamma=|dz|^2$ then $\gamma_{zz}=\gamma_{\bar z\bar z}=0$ $\gamma_{z\bar z}=\frac{1}{2}$ $\gamma^{z\bar z}=2$, $R=0$

$$\langle T_{z\bar z}\rangle = 0$$

Exercise Include primary fields in the correlator

If $\sigma=0$ around the insertion point, then

cor. do not change.

γ is called locally flat if it is of the form $|dz|^2$ around insertions, drop subscript γ.

As an example:

$$\langle T_{zz}\rangle_{e^{\sigma}\gamma} = \langle T_{zz}\rangle_{\gamma} + \frac{c}{24}\frac{\delta}{\delta\gamma^{zz}}\Big(\|d\sigma\|^2 + 4\int_\Sigma \sigma R\,dv\Big)$$

$$\langle T_{zz}\rangle^{\sigma}_{e\gamma} = Z^{-1}_{e^{\sigma}\gamma}\frac{4\pi\delta}{\delta e^{\sigma}\gamma^{zz}} \underset{\text{expand}}{Z_{e^{\sigma}\gamma}}\Big) \quad \sigma = 0 \text{ in the ins. point}$$

<u>Lemma</u> Let $\gamma^{z\bar z} = \gamma^{\bar z z} = 2$. To the first order in γ^{zz}

$$R = -\frac{1}{2}\left(\partial_z^2\gamma^{zz} + \partial_{\bar z}^2\gamma^{\bar z\bar z}\right)$$

<u>Proof:</u> $\gamma^{-1} = \gamma^{zz}\partial_z^2 + 4\partial_z\partial_{\bar z} + \gamma^{\bar z\bar z}\partial_{\bar z}^2$

We want to reduce metric to the form $(4+\rho)\partial_{z'}\partial_{\bar z'}$

$z' = z + \zeta(z,\bar z)$ - Diff.

$\partial_z = (1+\partial_z\zeta)\partial_{z'} + (\partial_z\bar\zeta)\partial_{\bar z'}$, $\partial_{\bar z} = (\partial_{\bar z}\zeta)\partial_{z'} + (1+\partial_{\bar z}\bar\zeta)\partial_{\bar z'}$

$$\gamma^{-1} = (\gamma^{zz} - (\partial_z\gamma^{zz})\zeta - (\partial_{\bar z}\gamma^{zz})\bar\zeta + 2\gamma^{zz}\partial_z\zeta + 4\partial_{\bar z}\zeta)\partial_{z'}^2$$

$$+ (4 + 4\partial_z\zeta + 4\partial_z\bar\zeta + 2\gamma^{zz}\partial_z\bar\zeta + 2\gamma^{\bar z\bar z}\partial_{\bar z}\zeta)\partial_{z'}\partial_{\bar z'}$$

$$+ \dots \qquad R\,v = \frac{i}{4}\bar\partial\partial(4+\rho)$$

Our req. $\gamma^{z'z'} = 0 \quad \partial_{\bar z}\zeta = -\frac{1}{4}\gamma^{zz}$

$$R'v' = -i\bar\partial'\partial'\log(1+\partial_z\zeta+\partial_{\bar z}\bar\zeta) =$$

$$= i\partial_{\bar z}\partial_z(\partial_z\zeta + \partial_{\bar z}\bar\zeta)\,dz\wedge d\bar z =$$

$$= -\frac{1}{2}\left(\partial_z^2\gamma^{zz} + \partial_{\bar z}^2\gamma^{\bar z\bar z}\right)v'$$

$$\frac{\delta}{\delta\gamma^{zz}}\Big(\|d\sigma\|^2 + 4\int_\Sigma \sigma R\,dv\Big) = -2\partial_z^2\sigma + (\partial_z\sigma)^2$$

$$\langle T_{zz}\rangle_{e^{\sigma}dz\,d\bar z} = \langle T_{zz}\rangle - \frac{c}{12}\Big(\partial_z^2\sigma - \frac{1}{2}(\partial_z\sigma)^2\Big)$$

W.r.t. holomorphic change of coordinates:

$$\left(\frac{dz'}{dz}\right)^2 \langle T_{z'z'}\rangle = \langle T_{zz}\rangle \left|\frac{dz'}{dz}\right|^2 dz\,d\bar{z} =$$

$$= \langle T_{zz}\rangle - \frac{c}{12}\left(\partial_z^2 \log\left(\frac{dz'}{dz}\right) - \frac{1}{2}\left(\partial_z \log\left(\frac{dz'}{dz}\right)\right)^2\right) =$$

$$= \langle T_{zz}\rangle - \frac{c}{12}\left(\frac{d^3z'/dz^3}{dz'/dz} - \frac{3}{2}\left(\frac{d^2z'/dz^2}{dz'/dz}\right)^2\right) = \langle T_{zz}\rangle - \frac{c}{12}\{z';z\}$$

T_{zz} transforms like a projective connection

Exercise 1) Show that

$$\partial_z^2 - \mathcal{U}(z) : \Omega_\Sigma^{-1/2} \to \Omega_\Sigma^{3/2}$$

$\mathcal{U}$ - transf. gives Schwarzian

2) Show that there is a bijection between proj. connections and proj structures

proj structure on Σ: $z_\beta = f_{\alpha\beta}(z_\alpha)$ fract. linear

two proj structures are eq. if union is also a proj. structure