Lecture III

Look at this expression again:

$$4\pi Z_\gamma^{-1} \frac{\delta}{\delta\delta}\Big|_{\delta=0} Z_{e^\delta\gamma} = -\gamma^{zz}\langle T_{zz}\rangle_\gamma - 2\gamma^{z\bar z}\langle T_{z\bar z}\rangle_\gamma - \gamma^{\bar z\bar z}\langle T_{\bar z\bar z}\rangle = \frac{c}{6}R$$

$$R \approx -\frac{1}{2}\left(\partial_z^2\gamma^{zz} + \partial_{\bar z}^2\gamma^{\bar z\bar z}\right)$$

Apply $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\gamma^{ww}} Z_\gamma$ at γ - locally flat

$$-\pi\,\delta^{(2)}(z-w)\langle T_{zz}\rangle - \langle T_{ww}T_{\bar z\bar z}\rangle = -\frac{\pi c}{12}\partial_{\bar z}^2\delta^{(2)}(z-w)$$

One can obtain further relations by means of derivations w.r.t. metric.

Diffeomorphism covariance:

$$f(z) = z + \zeta(z,\bar z) \equiv z'$$

$$\delta\gamma^{zz} = -(\partial_z\gamma^{zz})\zeta - (\partial_{\bar z}\gamma^{zz})\bar\zeta + 2\gamma^{zz}\partial_z\zeta + 4\partial_{\bar z}\zeta$$

$$\delta\gamma^{z\bar z} = 2\partial_z\zeta + 2\partial_{\bar z}\bar\zeta + \gamma^{zz}\partial_z\bar\zeta + \gamma^{\bar z\bar z}\partial_{\bar z}\zeta$$

$$\delta\gamma^{\bar z\bar z} = -(\partial_{\bar z}\gamma^{\bar z\bar z})\bar\zeta - (\partial_z\gamma^{\bar z\bar z})\zeta + 2\gamma^{\bar z\bar z}\partial_{\bar z}\bar\zeta + 4\partial_z\bar\zeta$$

$$\int_\Sigma \left(\langle T_{zz}\rangle_\gamma\,\delta\gamma^{zz} + 2\langle T_{z\bar z}\rangle_\gamma\,\delta\gamma^{z\bar z} + \langle T_{\bar z\bar z}\rangle\,\delta\gamma^{\bar z\bar z}\right)du = 0$$

Looking at ζ - terms and keeping first order terms in $\gamma^{zz}, \gamma^{\bar z\bar z}$

$$-(\partial_z \gamma^{zz})\langle T_{zz}\rangle_\gamma - 2\partial_z(\gamma^{zz}\langle T_{zz}\rangle_\gamma)$$
$$-4\partial_{\bar z}\langle T_{zz}\rangle_\gamma - 4\partial_z\langle T_{z\bar z}\rangle_\gamma - 2\partial_{\bar z}(\gamma^{\bar z\bar z}\langle T_{\bar z\bar z}\rangle) \qquad (*)$$
$$-(\partial_z\gamma^{\bar z\bar z})\langle T_{\bar z\bar z}\rangle = 0$$

Setting $\gamma^{zz}=0 \Rightarrow \boxed{\partial_{\bar z}\langle T_{zz}\rangle = 0 = \partial_z\langle T_{\bar z\bar z}\rangle}$

Therefore T_{zz} ($T_{\bar z\bar z}$) are (anti) analytic away from insertions.

Let us apply $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\gamma^{ww}}Z_\gamma$ to $(*)$ at γ locally flat

$$\pi\left(\partial_z\delta^{(2)}(z-w)\right)\langle T_{zz}\rangle + 2\pi\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle +$$
$$+\partial_{\bar z}\langle T_{zz}T_{ww}\rangle + \partial_z\langle T_{z\bar z}T_{ww}\rangle = 0$$

this term we've seen before

$\partial_z\langle T_{ww}T_{z\bar z}\rangle =$ (from previous formula we derived in the beginning)

$$= \frac{\pi c}{12}\partial_z^3\delta^{(2)}(z-w) - \partial_z\left(\delta^{(2)}(z-w)\langle T_{zz}\rangle\right)$$

Therefore,

$$\partial_{\bar z}\langle T_{zz}T_{ww}\rangle = -\pi\left(\partial_z\delta^{(2)}(z-w)\right)\langle T_{zz}\rangle -$$
$$-\frac{\pi c}{12}\partial_z^3\delta^{(2)}(z-w) - \pi\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle =$$
$$= -\frac{\pi c}{12}\partial_z^3\delta^{(2)}(z-w) - 2\pi\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle + \pi\delta^{(2)}(z-w)\partial_w\langle T_{ww}\rangle$$

$$\delta^{(2)}(z-w) = \frac{1}{\pi}\partial_{\bar z}\frac{1}{z-w} \Rightarrow$$

$$\partial_{\bar z} \langle T_{zz} T_{ww} \rangle = \partial_{\bar z} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} \langle T_{ww} \rangle + \frac{1}{z-w} \partial_w \langle T_{ww} \rangle \right)$$

Therefore,

$$\langle T_{zz} T_{ww} \rangle = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} \langle T_{ww} \rangle + \frac{1}{z-w} \partial_w \langle T_{ww} \rangle + \dots$$

... functions, analytic in z around $z=w$

First example of Operator product expansion

<u>Mixed insertions:</u> $\frac{\pi}{Z_\gamma} \frac{\delta}{\delta \gamma^{\bar w \bar w}} Z_\gamma$ applied to (*)

$$\partial_{\bar z} \langle T_{zz} T_{\bar w \bar w} \rangle + \partial_z \langle T_{z\bar z} T_{\bar w \bar w} \rangle + \pi \left(\partial_z \delta^{(2)}(z-w) \right) \langle T_{\bar z \bar z} \rangle = 0$$

eliminate using the first relation

$$\partial_{\bar z} \langle T_{zz} T_{\bar w \bar w} \rangle = -\frac{\pi c}{12} \partial_z \partial_{\bar z}^2 \delta^{(2)}(z-w) \Rightarrow$$

$$\Rightarrow \langle T_{zz} T_{\bar w \bar w} \rangle = -\frac{\pi c}{12} \partial_z \partial_{\bar z} \delta^{(2)}(z-w) + \text{funct. analytic in } z \text{ and } \bar w.$$

contact term

<u>Primary field insertions</u>

Let us apply $\frac{\pi}{Z_\gamma} \frac{\delta}{\delta \delta}\Big|_{\delta=0} Z e^{\delta \gamma} \langle \phi_e(x) \rangle e^{\delta \gamma}$ and

γ - locally flat

$$\langle T_{z\bar z} \, \phi_e(w, \bar w) \rangle = \pi \Delta_e \delta^{(2)}(z-w) \langle \phi_e(w, \bar w) \rangle$$

Diffeomorphisms:

$$f(z) = z + \zeta(z,\bar z) \equiv z' \qquad \gamma = f^{*}\gamma'$$

$$\langle \phi_e(w',\bar w')\rangle_{\gamma'} Z_{\gamma'} = \langle \phi_e(w,\bar w)\rangle_{\gamma} Z_{\gamma}$$

If $\gamma = |dz|^2$

$$\delta\gamma^{-1} = (\gamma')^{-1} - \gamma^{-1} = 4(\partial_{\bar z}\zeta)\partial_z^2 + 4(\partial_z\zeta + \partial_{\bar z}\bar\zeta)\partial_z\partial_{\bar z} + 4(\partial_z\bar\zeta)\partial_{\bar z}^2$$

first order in ζ:

$$\pi\,\delta^{(2)}(z-w)\,\partial_w\langle\phi_e(w,\bar w)\rangle - \partial_{\bar z}\langle T_{zz}\,\phi_e(w,\bar w)\rangle - \partial_z\langle T_{z\bar z}\,\phi_e(w,\bar w)\rangle = 0$$

We use previous eq. to eliminate

$$\partial_{\bar z}\langle T_{zz}\,\phi_e(w,\bar w)\rangle = -\pi\,\Delta_e\,\delta^{(2)}(z-w)\langle\phi_e(w,\bar w)\rangle + \pi\,\delta^{(2)}(z-w)\,\partial_w\langle\phi_e(w,\bar w)\rangle$$

Therefore, we have:

$$\langle T_{zz}\,\phi_e(w,\bar w)\rangle = \left(\frac{\Delta_e}{(z-w)^2} + \frac{1}{(z-w)}\partial_w\right)\langle\phi_e(w,\bar w)\rangle + \dots$$

<u>Note:</u>
$$\langle\phi_e(z',\bar z')\rangle = \langle\phi_e(z,\bar z)\rangle_{|dz'/dz|^2 dz d\bar z} = \left|\frac{dz'}{dz}\right|^{-2\Delta_e}\langle\phi_e(z,\bar z)\rangle$$

or
$$\langle\phi_e(z',\bar z')\rangle(dz')^{\Delta_e}(d\bar z')^{\Delta_e} = \langle\phi_e(z,\bar z)\rangle(dz)^{\Delta_e}(d\bar z)^{\Delta_e}$$

ϕ_e is a (Δ_e,Δ_e)-form

In the following we will consider primary fields $(\Delta_e,\tilde\Delta_e)$ s.t. $\Delta_e - \tilde\Delta_e \in \mathbb{Z}$

We call $d_e = \Delta_e + \bar{\Delta}_e$ - scaling dimension
and $s_e = \Delta_e - \bar{\Delta}_e$ - spin

Results of today's lecture

1) $T_{z\bar{z}} = T_{\bar{z}z} = 0$ $\quad \partial_{\bar{z}} T = \partial_z \bar{T} = 0$ $\quad T, \bar{T}$ - notations for $T_{zz}, T_{\bar{z}\bar{z}}$

2) OPEs:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \dots$$

$$T(z)\bar{T}(w) = -\frac{\pi c}{12}\partial_z \partial_{\bar{z}} \delta^{(2)}(z-w) + \dots$$

$$T(z)\phi_e(w,\bar{w}) = \left(\frac{\Delta_e}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)\phi_e(w,\bar{w}) + \dots$$

3) Transformation laws:

$$T(z')(dz')^2 = T(z)(dz)^2 - \frac{c}{12}\{z';z\}\,dz^2$$

$$\phi_e(z',\bar{z}')\,(dz')^{\Delta_e}(d\bar{z}')^{\bar{\Delta}_e} = \phi_e(z,\bar{z})(dz)^{\Delta_e}(d\bar{z})^{\bar{\Delta}_e}$$