Lecture IV

Space of states? Symmetries?

Consider discs D & D' $\quad D \equiv \{|z| \le 1\}$ $\quad D' \equiv \{|z| \ge 1\}$

$\Theta : \mathbb{CP}^1 \to \mathbb{CP}^1 \quad \Theta(z) = \bar{z}^{-1}$ – interchanges D and D'

γ – metric on D, s.t. it has the form $|z|^{-2}|dz|^2$ around ∂D. $\Theta^*\gamma$ is a metric on D' $\Rightarrow$

$\Rightarrow \Theta^*\gamma \vee \gamma$ is a metric on $\mathbb{CP}^1$

Formal expression: $X = \prod_i \phi_{e_i}(z_i, \bar{z}_i)$, $z_i \neq z_j$

Let $\Theta \phi_e(z,\bar{z}) \equiv (-\bar{z}^{-2})^{\Delta_e} (-z^{-2})^{\tilde{\Delta}_e} \phi_{\bar{e}}\left(\frac{1}{\bar{z}}, \frac{1}{z}\right)$

where $e \to \bar{e}$ is an involution we saw in the "constructive" axioms.

$$\Theta X = \prod_i \Theta \phi_{e_i}(z_i, \bar{z}_i)$$

Physical positivity: $\quad \forall \lambda_\alpha \in \mathbb{C} \quad \forall \gamma_\alpha$ on D flat at the boundary

$$\sum_{\alpha_1, \alpha_2} \bar{\lambda}_{\alpha_2} \lambda_{\alpha_1} Z_{\Theta^*\gamma_{\alpha_2} \vee \gamma_{\alpha_1}} \langle (\Theta X_{\alpha_2}) X_{\alpha_1} \rangle_{\Theta^*\gamma_{\alpha_2} \vee \gamma_{\alpha_1}} \ge 0$$

$$\Theta T(z) = \bar{z}^{-4}\, T\left(\frac{1}{\bar{z}}\right)$$

$$\Theta \bar{T}(\bar{z}) = z^{-4}\, \bar{T}\left(\frac{1}{z}\right)$$

$$Y = \prod_m T(z_m) \prod_n \bar{T}(\bar{z}_n) \prod_i \phi_{e_i}(z_i, \bar{z}_i)$$

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$$\Theta Y = \prod_m \Theta T(z_m) \prod_n \Theta \bar{T}(\bar{z}_n) \prod_i \Theta \phi_{e_i}(z_i, \bar{z}_i)$$

<u>Ex.</u> Prove that $\sum_{\alpha_1, \alpha_2} \bar{\lambda}_{\alpha_2} \lambda_{\alpha_1} \langle (\Theta Y_{\alpha_2}) Y_{\alpha_1} \rangle \geq 0$ in locally flat metric.

V_D – space of formal lin. combinations of Y's

$\mathcal{H} = \overline{V_D / V_D^{null}}$ w.r.t. to defined Hermitean form.

$i : V_D \to \mathcal{H}$ $\quad \mathcal{H}_0 = \mathrm{Im}\, i \quad i(Y) = y$

$\Omega = i(1)$ – vacuum vector

<u>$\mathcal{H}$ has an anti-unitary involution:</u> (denote $\mathcal{I}$)

$$\bar{Y} = \prod_m T(\bar{z}_m) \prod_n \bar{T}(z_n) \prod_e \phi_{e_i}(\bar{z}_i, z_i)$$

Virasoro algebra from EM tensor

Dilations: let $q \in \mathbb{C}$, $0 < |q| \leq 1$

<u>Define:</u) $S_q T(z) = q^2 T(qz)$, $S_q \bar{T}(\bar{z}) = \bar{q}^2 T(\bar{q}\bar{z})$,

$$S_q \phi_e(z, \bar{z}) = q^{\Delta e} \bar{q}^{\Delta e} \phi_e(qz, \bar{q}\bar{z})$$

Easy to see that:

$$\langle (\Theta Y') S_q Y \rangle = \langle (\Theta S_{\bar{q}} Y') Y \rangle$$

use diff. invariance.

On the level of Hilbert space:

$S_q \mathcal{Y} = i(S_q Y)$

S_q defined on a dense inv. domain $\mathcal{H}_0$

Note semigroup property: $S_{q_1} S_{q_2} = S_{q_1 q_2}$

$|(\mathcal{Y}', S_q \mathcal{Y})| \le \|\mathcal{Y}'\| \, \|S_q \mathcal{Y}\| = \|\mathcal{Y}'\| (\mathcal{Y}, S_{\bar{q}q} \mathcal{Y})^{1/2} \le \ldots$

$\le \|\mathcal{Y}'\| \, \|\mathcal{Y}\|^{1/2 + \ldots + 1/2^{n+1}} (\mathcal{Y}, S_{(\bar{q}q)^{2^{n-1}}} \mathcal{Y})^{1/2^n}$

Assume that $\forall \varepsilon > 0 \ \exists \ C_\varepsilon$ s.t.

$|\langle (\Theta Y') S_t Y \rangle| \le C_\varepsilon t^{-\epsilon}$ when $t \to 0 \Rightarrow$

$\Rightarrow |(\mathcal{Y}', S_q \mathcal{Y})| \le \|\mathcal{Y}'\| \, \|\mathcal{Y}\| \Rightarrow S_q$ - is a contraction

Note $S_q^* = S_{\bar{q}}$. By abstract semigroup theory

$S_q = q^{L_0} \bar{q}^{\tilde{L}_0}$ for commuting $L_0, \tilde{L}_0$ $\quad L_0 + \tilde{L}_0 \ge 0$

$L_0 \mathcal{Y} = \partial_q |_{q=1} S_q \mathcal{Y}$, $\tilde{L}_0 \mathcal{Y} = \partial_{\bar{q}} |_{q=1} S_q \mathcal{Y}$

Also $S_q \mathcal{H}_0$ is dense in $\mathcal{H}$ $\forall q$.

Introduce operators: $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$, $\varphi_e(z, \bar{z})$

s.t. $S_z \mathcal{H}_0$ is a dense domain:

$\mathcal{T}(z) \mathcal{Y} = i(T(z) Y)$ $\quad \bar{\mathcal{T}}(\bar{z}) \mathcal{Y} = i(\bar{T}(\bar{z}) Y)$

$\varphi_e(z, \bar{z}) \mathcal{Y} = i(\phi_e(z, \bar{z}) Y)$

Note $\mathcal{Y} = R\left(\prod_m T(z_m) \prod_n \bar{T}(\bar{z}_n) \prod_i \varphi_{e_i}(z_i, \bar{z}_i)\right) \Omega$

Here $R(\ldots)$ reorders the operators so that they act in the order of increasing $|z|$. (radial quantization)

Ex. Show that under I (anti-unitary inv.)

$$I\varphi_e(z,\bar z)I = \varphi_{\bar e}(\bar z, z)$$

$$I\,T(z)\,I = T(\bar z) \qquad I\,\bar T(\bar z)\,I = T(\bar z)$$

Define Fourier components:

$$L_n = \frac{1}{2\pi i}\oint_{|z|=r<1} z^{n+1}T(z)\,dz, \quad \bar L_n = \frac{-1}{2\pi i}\oint_{|\bar z|=r<1} \bar z^{n+1}\bar T(\bar z)\,d\bar z$$

Inside the matrix element $(\Psi', L_n\Psi)$ we are free to deform the contour, as long as it encircles insertions of Ψ.

Proposition $\quad L_n^* = L_{-n}$

Proof (next page)

$$(y', L_n y) = \frac{1}{2\pi i} \oint_{|z|=1-\epsilon} z^{n+1} \langle (\Theta Y') T(z) Y \rangle dz =$$

$$= \frac{1}{2\pi i} \oint_{|z|=1+\epsilon} z^{n-3} \langle (\Theta (Y' T(\tfrac{1}{z}))) Y \rangle dz$$

$T(z)$ with $|z| = 1+\epsilon$ $\quad z^{-4} \Theta T(\frac{1}{z})$

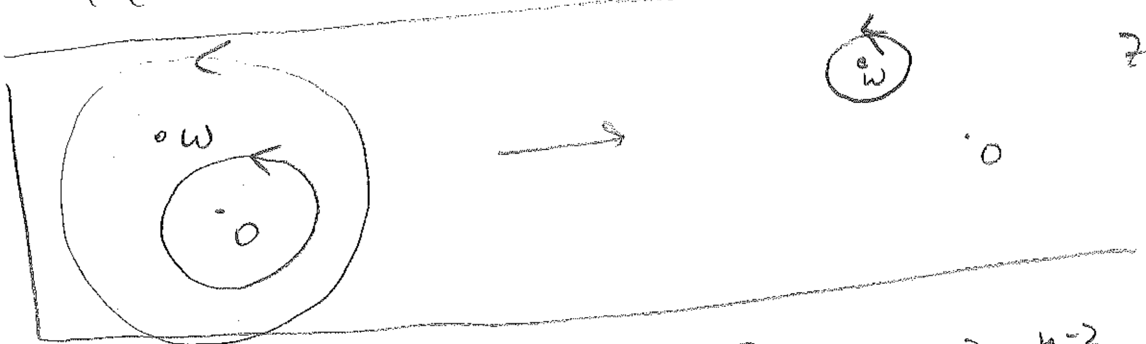
$$\left(-\frac{1}{2\pi i} \oint_{|z|=1+\epsilon} \bar{z}^{n-3} \overset{\text{operator}}{T_{zz}}\left(\frac{1}{z}\right) dz\, y', y \right) =$$

$$= \left(\frac{1}{2\pi i} \oint_{|w|=(1+\epsilon)^{-1}} w^{-n+1} T(w) dw\, y', y \right) = (L_{-n} y', y)$$

Commutator and Virasoro algebra

$$(y', [L_n, T(w)] y) = \frac{1}{2\pi i} \left(\oint_{|z|=|w|+\epsilon} dz - \oint_{|z|=|w|-\epsilon} dz \right) z^{n+1} \langle (\Theta Y') T(z) T(w) Y \rangle$$

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) + \dots$$



z - integration

$$z^{n+1} = ((z-w)+w)^{n+1} = \frac{n^3-n}{6} (z-w)^3 w^{n-2} + \frac{n^2+n}{2} (z-w)^2 w^{n-1} + (n+1)(z-w) w^n + w^{n+1} + \dots$$

$$(y', [L_n, T(w)] y) = \langle (\Theta Y') \left(\frac{c}{12} (n^3-n) w^{n-2} + 2(n+1) w^n T(w) + w^{n+1} \partial_w T(w) \right) Y \rangle$$

$$[L_n, T(\omega)] = \frac{c}{12}(n^3-n)\omega^{n-2} + 2(n+1)\omega^n T(\omega) + \omega^{n+1}\partial_\omega T(\omega)$$

similarly for $\tilde{L}_n$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

$\ell_n = -z^{n+1}\partial_z$ – basis for vector fields on a circle

$$0 \to \mathbb{C} \to Vir \to Vect(S^1) \to 0$$

Virasoro algebra is a central extension.

<u>Exercise</u> $[L_n, \varphi_e(\omega,\bar\omega)] = \Delta_e(n+1)\omega^n\varphi_e(\omega,\bar\omega) + \omega^{n+1}\partial_\omega\varphi_e(\omega,\bar\omega)$

$[\tilde{L}_n, \varphi_e(\omega,\bar\omega)] = \bar\Delta_e \ldots$

One can prove (see Gawedzki) that
$L_0, \bar{L}_0$ have positive spectrum

Notice: $L_n\Omega = 0$, $n \geq -1$ $\qquad \tilde{L}_n\Omega = 0$, $n \geq -1$

$L_n\varphi_e(0)\Omega = 0$, $n > 0$ $\qquad \tilde{L}_n\varphi_e(0) = 0$ $n > 0$

$L_0\varphi_e(0)\Omega = \Delta_e\varphi_e(0)\Omega$, $\qquad \tilde{L}_0\varphi_e(0)\Omega = \tilde\Delta_e\varphi_e(0)\Omega$

$$\varphi_e(0)\Omega = \lim_{z\to 0}\varphi_e(z,\bar z)\Omega$$

Primary fields provide highest weight modules for Vir.