Lecture V

Back to Segal's axioms. Path integral interpretation

Σ - Riemann surface with holes C_i - boundaries

$i \in I_{in} \cup I_{out}$ $\quad$ γ_i is the metric on the corresponding "cut out" disk D.

$$\tilde{\gamma} = \left(\bigvee_{i \in I_{in}} \gamma_i\right) \vee \gamma \vee \left(\bigvee_{i \in I_{out}} \theta^* \gamma_i\right)$$

is the metric on $\hat{\Sigma}$ (surface with glued discs)

__Definition__

$$Z_{\tilde{\gamma}} \prod_{i \in I} \left(Z_{\theta^* \gamma_i \vee \gamma_i} \right)^{-1/2} \equiv Z_\gamma \text{ (partition fn on } \Sigma)$$

It is independent on the conformal factors of γ_i.

Transformation of Z_γ under $\gamma \to e^{\sigma}\gamma$ is given by the same equation.

__Define__: $\left(\bigotimes_{i \in I_{out}} \psi_i , A_{\Sigma,\gamma} \bigotimes_{i \in I_{in}} \psi_i \right) =$

$$= Z_\gamma \left\langle \prod_{i \in I_{out}} (\theta \Upsilon_i) \prod_{i \in I_{in}} \Upsilon_i \right\rangle$$

Path integral interpretation

ϕ – fields (sections of some vector bundles over Σ)

Measure $e^{-S_\Sigma(\phi)} D\phi$

$$Z_\gamma \equiv \int e^{-S_\Sigma(\phi)} D\phi \quad \text{on a closed } \Sigma$$

$$A_{\Sigma,\gamma}((\varphi_i)_{i\in I}) = \int_{\phi|_{C_i}=\varphi_i} e^{-S_\Sigma(\phi)} D\phi$$

It is a function of the field boundary values

Formal L^2 pairing: $G(\varphi), F(\varphi)$ – functionals

$\langle G, F\rangle = \int D\varphi\, \overline{G(\varphi)}\, F(\varphi)$ – defines space of states $\mathcal{H}$

$\mathcal{H}$ has anti-unitary involution I:

$I F(\varphi) = \overline{F(\varphi^\vee)}$ $\quad \varphi^\vee(z) = \varphi(z^{-1})$

Gluing axiom:

$$\int_{\substack{\phi|_{C_i}=\varphi_i,\\ i\neq i_0,i_1}} e^{-S_{\Sigma'}(\phi)} D\phi = \int \left(\int_{\substack{\phi|_{C_i}=\varphi_i,\, i\neq i_0,i_1\\ \phi|_{C_{i_0}}=\varphi_0=\phi|_{C_{i_1}}}} e^{-S_\Sigma(\phi)} D\phi \right) D\varphi_0$$

Σ' is obtained from Σ by gluing C_{i_0} an C_{i_1}

What about vector $\mathcal{Y} \in \mathcal{H}$?

$$\mathcal{F}_\Upsilon(\varphi) = \left(Z_{\theta^*\gamma \cup \gamma}\right)^{-1/2} \int_{\phi|_{\partial D} = \varphi} \Upsilon e^{-S_D(\phi)} D\phi$$

again φ is on the circle.

Complex conjugate:

$$\overline{\mathcal{F}_\Upsilon(\varphi)} = \left(Z_{\theta^*\gamma \cup \gamma}\right)^{-1/2} \int_{\phi|_{\partial D} = \varphi} \theta\Upsilon e^{-S_{D'}(\phi)} D\phi$$

i.e. $$\int \overline{\mathcal{F}_\Upsilon(\varphi)} \, \mathcal{F}_\Upsilon(\varphi) D\varphi = \left(Z_{\theta^*\gamma \cup \gamma}\right)^{-1} \int (\theta\Upsilon)\Upsilon e^{-S_{D \cup D'}(\phi)} D\phi$$

$$= \langle (\theta\Upsilon)\Upsilon \rangle = (\mathcal{Y}, \mathcal{Y})$$

Finally $\left(\otimes_{i \in I_{out}} \mathcal{Y}_i, A_{\Sigma,\gamma} \otimes_{i \in I_{in}} \mathcal{Y}_i \right) = Z_\gamma \langle \prod_{i \in I_{out}} (\theta \Upsilon_i) \prod_{i \in I_{in}} \Upsilon_i \rangle$

follows from

$$Z_{\tilde{\gamma}} \langle \prod_{i \in I_{out}} (\theta \Upsilon_i) \prod_{i \in I_{in}} \Upsilon_i \rangle =$$

$$= \int \prod_{i \in I_{out}} (\theta \Upsilon_i) \prod_{i \in I_{in}} \Upsilon_i \, e^{-S_{\hat{\Sigma}}(\phi)} D\phi$$

Exercise: show that! (check that all the factors find their places!

Let us have a look how Segal's axioms work.

1) Discs

Let $\Sigma = D$, then $(Z_{\theta^*\gamma \vee \gamma})^{-1/2} A_{D,\gamma}$ is a metric independent vector, vacuum

$$\underline{(Z_{\theta^*\gamma\vee\gamma})^{-1/2} A_{D,\gamma} \equiv \Omega}$$

$A_{D',\theta^*\gamma}$ is a dual element

$$\underline{(Z_{\theta^*\gamma\vee\gamma})^{-1/2} A_{D',\theta^*\gamma} = (\Omega, \cdot)}$$

$$\underline{\Omega = \mathbb{1}\,\Omega}$$

2) Annuli - semigroup, which encodes both Vir algebra action on $\mathcal{H}$

$$\Sigma = \{ |q| \le |z| \le 1 \} = C_q$$

. Out boundary is parametrized by $z \to z$

. In boundary —— " —— by $z \to qz$

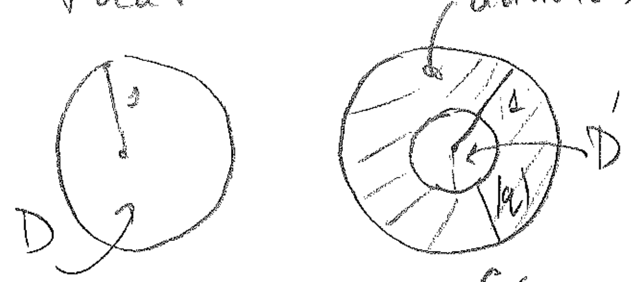
$$\gamma_0 = |z|^{-2} |dz|^2$$

$$(\psi', A_{C_q,\gamma_0}\, \psi) = Z_{\gamma_0} \langle (\Theta\Upsilon) S_q \Upsilon \rangle = Z_{\gamma_0} (\psi', S_q \psi)$$

$$Z_{\gamma_0} = \frac{Z_\gamma}{Z^{1/2}_{\gamma_{in}^* \vee \gamma_{in}}\, Z^{1/2}_{\gamma_{out} \vee \gamma^*_{out}}}$$

Notice that

$Z_{\tilde{\gamma}}$:



<u>Prove:</u> $Z_{\gamma_0} = (q\bar{q})^{-c/24}$

Therefore, $A_{C_q,\gamma_0} = q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24}$

$z \mapsto e^{iz}$ $\quad C_\tau = \{ z \mid 0 \le \operatorname{Im} z \le 2\pi\tau_2 \} / 2\pi\mathbb{Z}$

$q = e^{2\pi i \tau}$

with the metric $|dz|^2$

Gluing in and out sides we obtain

$T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with metric $4\pi^2 |dz|^2$

$Z(\tau) = A_{T_\tau, |dz|^2} = \operatorname{tr} q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24}$

by definition modular invariant

<u>toroidal</u>
partition function

<u>Other annular surfaces</u>

$f: D \to D$ - holomorphic embedding, preserving origin

$\Rightarrow \Sigma_f = D \setminus f(\operatorname{int}(D))$ - annulus

$f_{q,\alpha,n}(z) = q^{z\partial_z} e^{\alpha z^{n+1}\partial_z} z$ $\quad \forall\, n \ge 0 \;\; |q| < 1$

α - small

$A_{\Sigma_{f_{q,\alpha,n}},\gamma} = Z_\gamma\, e^{\alpha L_n} q^{L_0} e^{\bar{\alpha} \bar{L}_n} \bar{q}^{\bar{L}_0}$

This encodes $L_n, \bar{L}_n$ for $n>0$

complex conj annuli give $L_n, \bar{L}_n$ for $n<0$

If $\chi_e \in \mathcal{H}$ is a highest weight vector of conformal weight $(\Delta_e, \tilde{\Delta}_e) \Rightarrow$

$$Z_\gamma^{-1} A_{\Sigma_{f,\gamma}} \chi_e = \left(\frac{df(0)}{dz}\right)^{\Delta_e} \left(\frac{d\bar{f}(0)}{d\bar{z}}\right)^{\tilde{\Delta}_e} \chi_e$$

Each Vir. Heighest weight vector $\rightarrow$ primary field

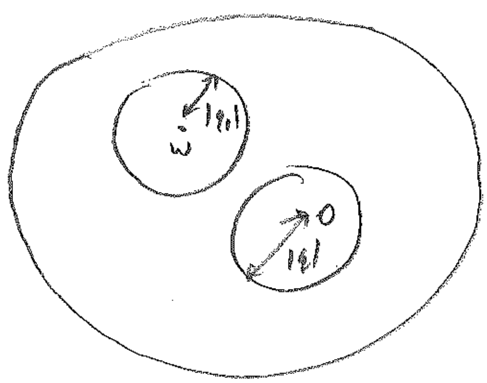
$$\langle \phi_{e_1}(x_1) \dots \phi_{e_n}(x_n) \rangle = \frac{1}{Z_\gamma} A_{\Sigma,\gamma} \otimes_i \chi_{e_i}$$

Pants, Vertex operators and associativity

$\chi \in \mathcal{H}$, s.t. $\chi \in$ domain of $q_1^{-L_0} \bar{q}_1^{-\bar{L}_0}$ where $|q_1| < 1 \rightarrow$ Vertex operator: $\varphi(\chi; w, \bar{w})$

defined for $0 < |w| < 1$

Let $0 < |q| < |w| - |q_1| < 1 - 2|q_1|$



$P_{q, q_1, w} = \{ |q| \le |z| \le 1, |z-w| \ge |q_1| \}$

"out" comp $|z| = 1$

"in" $z \mapsto qz$

$z \mapsto w + q_1 z$

Define: $\varphi(\chi; w, \bar{w}) Y = \frac{1}{Z_\gamma} A_{P_{q, q_1, w, \gamma}} (q^{-L_0} \bar{q}^{-\bar{L}_0} Y \otimes q_1^{-L_0} \bar{q}_1^{-\bar{L}_0} \chi)$

where $Y \in$ domain of $q^{-L_0} \bar{q}^{-\bar{L}_0}$

Exercises

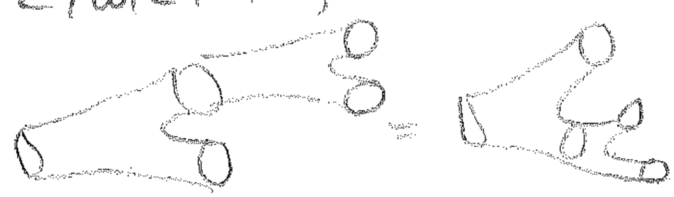
1) Show independence on q, q_1
2) Show that $\varphi(X;0)\Omega = \lim_{w\to 0} \varphi(X; w, \bar w)\Omega = X$
3) If X_e is a h.w. vector then $\varphi(X_e, w, \bar w)$ coincides with primary fied $\phi_e(w, \bar w)$
4) Show that $\varphi(L_{-2}\Omega, w, \bar w) = T(w)$

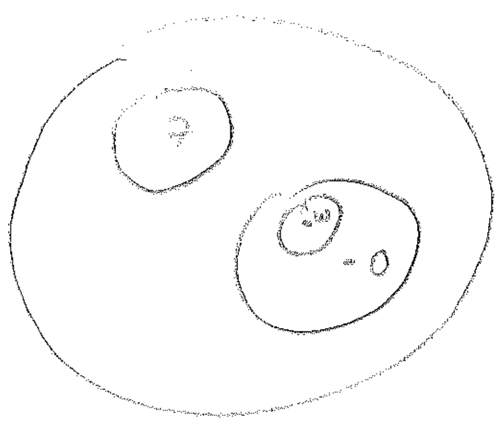
   $\varphi(\bar L_{-2}\Omega, w, \bar w) = \bar T(\bar w)$

Associativity property:

$$\varphi(Y; z, \bar z)\,\varphi(X; w, \bar w) = \varphi(\varphi(Y, z-w, \bar z - \bar w)X; w, \bar w)$$

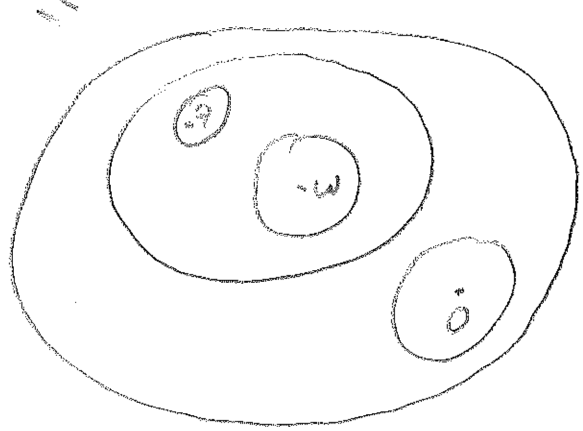
holding for $0<|w|<|z|$, $0<|z-w|<1$

<u>Proof</u> (sketch) 



$\varphi(Y, z, \bar z)\varphi(X, w, \bar w)$

=



$= \varphi(\varphi(Y, z-w, \bar z - \bar w)X; w, \bar w)$