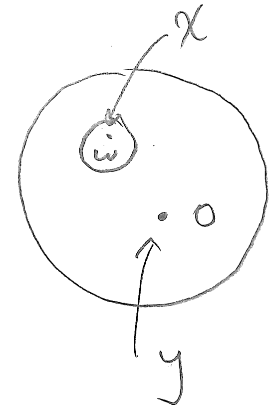
Lecture VI

Vertex operator algebras (VOA)

We have seen that CFT implies that there is a natural map:

$\mathcal{H}^{\otimes 2} \to \mathcal{H}$ associated with



We called it "vertex operator"

$$\varphi(x; w, \bar{w}) \cdot y$$

Assume that the dependence of φ on w is holomorphic only, i.e.

$$\partial_{\bar{w}} \varphi(x, w, \bar{w}) = 0$$ everywhere, except other punctures with insertions. There, the meromorphic structure of singularities will imply the equation above to be correct modulo δ-functions.

It is natural to think about $\varphi(x, w)$ as formal series $\displaystyle\sum_{n=-\infty}^{\infty} \frac{x_n}{z^{n+1}}$

On the level of formal series

$$[\varphi(x,w), \varphi(x,z)](z-w)^N = 0$$

is equivalent to the associativity property as we will see.

Before we proceed to the definition, let us introduce several notions of formal calculus.

Let R be a $\mathbb{C}$-algebra

<u>Def.</u> $R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$ is the vector space

spanned by $A(z_1, \dots, z_n) = \sum_{i_1 \in \mathbb{Z}} \dots \sum_{i_n \in \mathbb{Z}} A_{i_1 \dots i_n} z_1^{i_1} \dots z_n^{i_n}$

where $A_{i_1 \dots i_n} \in R$.

In general, product of two such terms does not make any sense, only when we deal with Laurent polyn.

<u>Delta-function</u>

$$\delta(z-w) = \sum_{m \in \mathbb{Z}} z^m w^{-m-1} \qquad a_{mn} = \delta_{m,-n-1}$$

$$A(w)\,\delta(z-w) = \sum A_k w^k \sum_m z^m w^{-m-1} = \sum A_{m+n+1} z^m w^n$$

$$\Downarrow$$

$$A(w)\,\delta(z-w) = A(z)\,\delta(z-w)$$

Also, $(z-w)\,\delta(z-w) = 0$ and $(z-w)^{n+1} \partial_w^n \delta(z-w) = 0$

Power series can be understood as distributions

$$f(z) = \sum_{i \in \mathbb{Z}} a_i z^i \qquad \operatorname{Res}_{z=0} f(z) = a_{-1}$$

Any formal power series defines a linear functional on the space of Laurent polynomials $\mathbb{C}[z, z^{-1}]$

$$\operatorname{Res}_{z=0} \underset{\mathbb{C}[[z,z^{-1}]]}{\underset{\cap}{f(z)}} \; \underset{\mathbb{C}[z,z^{-1}]}{\underset{\cap}{g(z)}} = \langle f, g \rangle$$

<u>Lemma</u> Let $f(z,w) \in R[[z^{\pm 1}, w^{\pm 1}]]$, satisfying

$(z-w)^N f(z,w) = 0$ for $N > 0 \Rightarrow$

$$f(z,w) = \sum_{i=0}^{N-1} g_i(w)\, \partial_w^i\, \delta(z-w), \quad g_i(w) \in R[[w^{\pm 1}]]$$

<u>Proof</u>: exercise. (see Frenkel-Ben-Zvi)

Thinking about $\delta(z-w)$ from analytic point of view:

$$\delta(z-w) = \underbrace{\frac{1}{z}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n}_{\delta_-(z-w)} + \underbrace{\frac{1}{z}\sum_{n>0}\left(\frac{z}{w}\right)^n}_{\delta_+(z-w)}$$

If $z, w \in \mathbb{C} \Rightarrow \delta_-(z-w) = \frac{1}{z-w}$, $\delta_+(z-w) = -\frac{1}{z-w}$

if $|z| < |w|$

If we talk about the distribution meaning

$$\delta(z-w)_\pm : \quad \mp \lim_{\epsilon \to +0} \int_{|z|=1} \frac{1}{z - wq^{\mp\epsilon}}\, g(z) \frac{dz}{2\pi i}$$

Similar to famous formula

$$2\pi i\, \delta(x) = \frac{1}{x+i0} - \frac{1}{x-i0}$$

Algebraic reformulation

$R[[z]]$ – formal Taylor series

$R((z))$ – formal Laurent series (field if R is a field)

Denote: $\mathbb{C}((z))((w)) = R((w))$, $R = \mathbb{C}((z))$

$$\delta(z-w)_- = \frac{1}{z}\sum_{n\geq 0} \frac{w^n}{z^n} \in \mathbb{C}[z^{-1}][[w]] \subset \mathbb{C}((z))((w))$$

this is algebraist way of saying $\delta(z-w)_-$ is exp. of $\frac{1}{z-w}$ in $|z|>|w|$

$$\delta(z-w)_+ = \frac{1}{z}\sum_{n>0} \frac{z^n}{w^n} \in \mathbb{C}((w))((z))$$

Denote: $\mathbb{C}((z,w))$ – field of fractions of $\mathbb{C}[[z,w]]$

Two completions of $\mathbb{C}((z,w))$

$$\begin{tikzcd}
\mathbb{C}((z))((w)) \arrow[dr, bend right] & \mathbb{C}((z,w)) \arrow[l, hook'] \arrow[r, hook] & \mathbb{C}((w))((z)) \arrow[dl, bend left] \\
& \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] &
\end{tikzcd}$$

$$\underset{|z|>|w|}{\delta(z-w)_-} \longleftarrow \frac{1}{z-w} \longrightarrow \underset{|z|<|w|}{-\delta(z-w)_+}$$

but images of $\mathbb{C}((z,w))$ elts are different

Completions are w.r.t. to topology of open neighborhoods of zero $w^N f(z,w)$, $N \in \mathbb{Z}$

f does not have neg. poles

this way we obtain $\mathbb{C}((z))((w))$

Note the intersection:

$$\mathbb{C}((z))((w)) \cap \mathbb{C}((w))((z)) \supset \mathbb{C}[[z,w]][z^{-1}, w^{-1}]$$

Local fields:

V - vector space over $\mathbb{C}$

$A(z) = \sum_j A_j z^{-j} \in \operatorname{End} V[[z^{\pm 1}]]$ is called <u>field</u> on V if $\forall v$ we have $A_j \cdot v = 0$ for j-large enough,

i.e. $A(z)v \in V((z))$

Let $V = \oplus_{n \in \mathbb{Z}} V_n$, $\phi : V \to V$ - <u>homogeneous of degree</u> <u>m</u> if $\phi(V_n) \subset V_{n+m}$

<u>Example</u> $[b_n, b_m] = n\delta_{n,-m}$

F - Fock module $V = \{ b_{-k_1} \dots b_{-k_n} |0\rangle \}$

$b_n |0\rangle = 0$, $n > 0$ $\quad$ $\deg = \sum k_i$

<u>Homogeneous field of conf dim $\Delta \in \mathbb{Z}$</u> is a field such that A_j is a homog. operator of degree $-j+\Delta$

<u>Rmks</u>: 1) If $V_k = 0$ $\forall k \leq K \Rightarrow A(z)$ is a field.

2) If $A(z)$ has conformal dim $\Delta \Rightarrow \partial_z A(z)$ has conf. dim $\Delta + 1$

<u>Example</u>: $b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$ - conf. dim 1.

In general $A(z) : V \to \overline{V}$ - completion of V if we take $z \in \mathbb{C}^\times$

Composition of fields

Let $v \in V$, $\varphi \in V^*$ - dual space,

then $\langle \varphi, A(z) B(w) v \rangle \in \mathbb{C}((z))((w))$

$\langle \varphi, B(w) A(z) v \rangle \in \mathbb{C}((w))((z))$

If we set them to commute, i.e. $\in \mathbb{C}[[w,z]][z^{-1}, w^{-1}]$
it is too strong

Def. $A(z)$, $B(w)$ are called local w.r.t each other if $\forall\, v \in V$, $\varphi \in V^*$, the matrix elements $\langle \varphi, A(z) B(w) v \rangle$ and $\langle \varphi, B(w) A(z) v \rangle$ are expansions of the same element $f_{v,\varphi} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ respectively
in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$
and pole in $(z-w)$ is uniformly bounded for v, φ.

Pole uniformly bounded $\Rightarrow \exists\, N \in \mathbb{Z}_+$, s.t.

$$(z-w)^N f_{v,\varphi} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}] \quad \forall\, v, \varphi$$

$$\Rightarrow \quad (z-w)^N [A(z), B(w)] = 0$$

Proposition Two fields $A(z)$, $B(w)$ are local w.r.t. each other iff $(z-w)^N [A(z), B(w)] = 0$

If V is $\mathbb{Z}$-graded (bounded from below) $\bigoplus_{n=k}^{\infty} V_n$

$V^\vee = \bigoplus_{n=k}^{\infty} V_n^*$ - restricted dual

If $\varphi \in V^\vee \Rightarrow \langle \varphi, A(z) v \rangle \in \mathbb{C}[z^{\pm 1}]$

$\langle \varphi, A(z) B(w) v \rangle \in \mathbb{C}[z^{\pm 1}]((w))$

Locality: $\langle \varphi, A(z) B(w) v \rangle$, $\langle \varphi, B(w) A(z) v \rangle$

are $f_{v,\varphi} \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{-1}]$

expansions in $\mathbb{C}[z^{\pm 1}]((w))$ and $\mathbb{C}[w^{\pm 1}]((z))$

with $(z-w)$-pole uniformly bounded

In other words if N is large enough,

$(z-w)^N \langle \varphi, A(z) B(w) v \rangle$ and $(z-w)^N \langle \varphi, B(w) A(z) v \rangle$

are expansions of the same polynomial

from $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$

Analytically:

$$P: V \to \bar{V} \Rightarrow P_i^j : V_i \hookrightarrow V \to \bar{V} \twoheadrightarrow V_j$$

$$(P \circ Q)_i^j = \sum_{k \in \mathbb{Z}_+} P_k^j Q_i^k$$

We say $P \circ Q$ exists if $\forall v, \varphi$ $\sum_{k \in \mathbb{Z}_+} \langle \varphi, P_k^j Q_i^k v \rangle$ conv. absolutely.

Then $P \circ Q : V \to \bar{V}$ is well-defined when

$(z-w)^N \langle \varphi, A(z) B(w) v \rangle$ is a polynomial.

<u>Proposition</u> $A(z), B(w)$ are local w.r.t. each other iff.

i) $\forall z, w \in \mathbb{C}^\times$ with $|z| > |w|$ the composition $A(z) B(w)$ exists, and can be analytically continued to an operator-valued meromorphic function $R(A(z) B(w))$ on $\mathbb{C}^2$ with sing. $\{z=0\}, \{w=0\}$ and $\{z=w\}$, s.t. the order of the pole at $z=w$ is unif. bounded

ii) For $|w| > |z|$ same for $B(w) A(z)$ $|w| > |z|$

iii) $R(A(z) B(w)) = R(B(w) A(z))$

<u>Exercise:</u> prove that $b(z), b(w)$ are mutually local.

Definition Vertex algebra:

1) V - space of states
2) $|0\rangle \in V$
3) $T : V \to V$
4) $Y(\cdot, z) : V \to \mathrm{End}\, V[[z^{\pm 1}]]$
s.t. $Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$
5) $Y(|0\rangle, z) = \mathrm{Id}_V$, $Y(A, z)|0\rangle \in V[[z]]$
and $Y(A, z)|0\rangle|_{z=0} = A$ $\quad (A_{(n)}|0\rangle = 0 \ n \geq 0$
$A_{(-1)}|0\rangle = A)$

6) $[T, Y(A, z)] = \partial_z Y(A, z)$
$T|0\rangle = 0$
7) All $Y(A, z)$ are local w.r.t each other

VA is called $\mathbb{Z}$-graded if V is a $\mathbb{Z}$-graded space, $|0\rangle$ has degree 0, T is a lin. operator of degree 1, $\forall A \in V_m$ $Y(A, z)$ has conf. dim m.