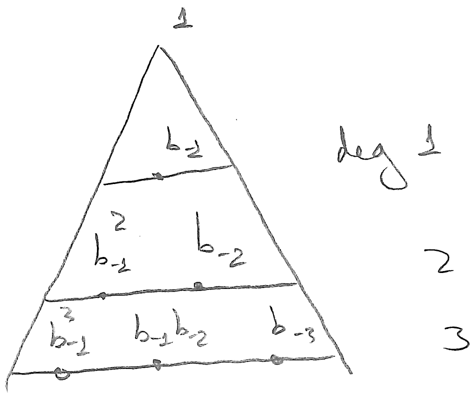
# Lecture VII Example: Heisenberg VOA

$\mathcal{H}$ $\quad 0 \to \mathbb{C}1 \to \mathcal{H} \to \mathbb{C}[t, t^{-1}] \to 0$

$b_n, n \in \mathbb{Z}$ $\quad [b_n, b_m] = n\delta_{n,-m}$

Obvious representation:

$F \simeq \mathbb{C}[b_{-1}, b_{-2}, \dots]$ $\quad b_n = n\dfrac{\partial}{\partial b_{-n}}$




Vertex algebra construction:

1) vacuum vector $|0\rangle = 1$

2) $T$ : $\quad [T, b_i] = -i\, b_{i-1}$, $T\cdot 1 = 0$

3) $\mathbb{Z}_+$-grading $\quad b_{j_1} \dots b_{j_k}$ $\quad j_1 \le j_2 \dots \le j_k < 0$ $\quad \deg = -\sum j_i$

Defining vertex operators:

$b_{-1}|0\rangle \to Y(b_{-1}|0\rangle, z) = b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$

$Y(b_{-k}|0\rangle, z) = \dfrac{1}{(k-1)!}\partial_z^{k-1} b(z)$ - easy to check

that $\lim_{z\to 0} Y(b_{-k}|0\rangle, z) = b_{-k}|0\rangle$

multilinear expressions?

$(b(z))^2$ - not well defined, so what about $b_{-1}^2|0\rangle$?

Normal ordering

$$:b(z)b(z): = \sum_{\substack{n\in\mathbb{Z}\\ n=k+l}} :b_k b_l:\, z^{-n-2}$$

$$:b_k b_l: = \begin{cases} b_l b_k & \text{if } l=-k,\ k\geq 0 \\ b_k b_l & \text{otherwise}\end{cases}$$

$z^{-2}$ coefficent : $\sum_{k\in\mathbb{Z}} :b_k b_{-k}: = 2\sum_{k<0} -k b_k \frac{\partial}{\partial b_k}$

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Def. In general: $A(z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$, $B(w) = \sum_{m\in\mathbb{Z}} B_{(m)} w^{-m-1}$

$$:A(z)B(w): = \sum_{n\in\mathbb{Z}} \Big( \sum_{m<0} A_{(m)} B_{(n)} z^{-m-1} + \sum_{m\geq 0} B_{(n)} A_{(m)} z^{-m-1}\Big) w^{-n-1}$$

$$= A(z)_+ B(w) + B(w) A(z)_-$$

Here $f(z)_+ = \sum_{n\geq 0} f_n z^n \quad f(z)_- = \sum_{n<0} f_n z^n$

Lemma i) $\forall v\in V$, $\varphi\in V^*$ the matrix element $\langle \varphi, :A(z)B(w): v\rangle$ lies in $\mathbb{C}[[z,w]][z^{-1},w^{-1}]$; $\forall z,w\in\mathbb{C}^\times$ is a well-def. op. $V\to\bar{V}$

ii) $:A(z)B(w):$ is well-defined for $z=w$. If $A(z)$ and $B(w)$ are homog. $\Delta_A$ and $\Delta_B$ $\Rightarrow$ $:A(z)B(w):$ has dim $\Delta_A+\Delta_B$

iii) $:A(w)B(w): = \mathrm{Res}_{z=0}\big(\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\big)$

In general:

$$:A(z)\,B(z)\,C(z): \;=\; :A(z):B(z)\,C(z)::$$

So, general VOAs:

$$Y(b_{j_1}\dots b_{j_k}, z) = \frac{1}{(-j_1-1)!\dots(-j_k-1)!} :\partial_z^{-j_1-1} b(z) \dots \partial_z^{-j_k-1} b(z):$$

<u>Locality for $b(z)$:</u>

$$b(z)\,b(w) = \sum_{n,m} b_n b_m z^{-n-1} w^{-m-1} =$$

$$= \underbrace{\sum_{N\in\mathbb{Z}/\{0\}} \sum_{n+m=N} b_n b_m z^{-n-1} w^{-m-1}}_{\Sigma} + \underbrace{\sum_{n\in\mathbb{Z}} b_{-n} b_n z^{n-1} w^{-n-1}}_{\Sigma_0} =$$

$$\Sigma_0 = :\Sigma_0: + (\Sigma_0 - :\Sigma_0:)$$

$$:\sum_{n\geq 0} b_{-n} b_n z^{n-1} w^{-n-1}: = \sum_{n\geq 0} b_{-n} b_n z^{n-1} w^{-n-1}$$

$$:\sum_{n<0} b_{-n} b_n z^{n-1} w^{-n-1}: = \sum_{n<0} b_n b_{-n} z^{n-1} w^{-n-1}$$

$$\Sigma_0 = :\Sigma_0: + \sum_{n<0} [b_{-n}, b_n] z^{n-1} w^{-n-1} = :\Sigma_0: + \sum_{m>0} m z^{-m-1} w^{m-1}$$

$$b(z)\,b(w) = :b(z)\,b(w): + \sum_{m>0} m z^{-m-1} w^{m-1}$$

expansion of $\frac{1}{(z-w)^2}$ in $\mathbb{C}((z))((w))$ $|z|>|w|$

$$b(z)\,b(w) = :b(z)\,b(w): + \frac{1}{(z-w)^2}$$

Similarly $b(w)\,b(z) = \frac{1}{(w-z)^2} + :b(w)\,b(z):$

Notice that $: \;:$ is comm. in this case.

Therefore:

$[b(z), b(w)] = \partial_w \delta(z-w)$

or $(z-w)^2 [b(z), b(w)] = 0$

Analytically

$R(b(z)\ b(w)) = :b(z)\,b(w): + \frac{1}{(z-w)^2}$

converges in $|z|>|w|$

$R(b(w)\ b(z)) = :b(w)\,b(z): + \frac{1}{(w-z)^2}$

Dong's lemma $A(z), B(z), C(z)$ are mut. local $\Rightarrow$

$\Rightarrow \; :A(z)\,B(z):$ and $C(z)$ are mut. local.

Proof: exercise.

Using Dong's Lemma it is easy to show that $F$ has a Vertex algebra structure.