Lecture VIII

Dong's Lemma (Proof)

Let r be such that $\forall s \geq r$

$$(w-z)^s A(z)B(w) = (w-z)^s B(w)A(z)$$

$$(u-z)^s A(z)C(u) = (u-z)^s C(u)A(z)$$

$$(u-w)^s B(w)C(u) = (u-w)^s C(u)B(w)$$

We need: find N s.t.

$$(w-u)^N :A(w)B(w): C(u) = (w-u)^N C(u) :A(w)B(w):$$

we show that this is true.

$$(w-u)^N \left(\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\right) C(w) =$$

$$= (w-u)^N C(u) \left(\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\right)$$

Take $N = 3r \Rightarrow$ in the LHS

$$(w-u)^{3r} = (w-u)^r \sum_{s=0}^{2r} \binom{2r}{s} (w-z)^s (z-u)^{2r-s}$$

All terms with $r < s \leq 2r$ vanish. ($(z-w)$ kills $\delta(z-w)$).

All other terms have $(z-u)^m$ $m \geq r$ which allows to move $C(u)$ through $A(z)$ while we still have $w-u$ to the rth power.

Generators and relations

V – vector space, $|0\rangle \in V$, $T \in \mathrm{End}(V)$, S – count. ordered set

$\{a^\alpha\}_{\alpha \in S}$ – collection of vectors in V,

1) $\forall \alpha$ $a^\alpha(z)|0\rangle = a^\alpha + z(\ldots)$

2) $T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ $\forall \alpha$

3) All $a^\alpha(z)$ are mut. local

4) V has basis of vectors

$$a^{\alpha_1}_{(j_1)} \ldots a^{\alpha_m}_{(j_m)}|0\rangle$$

where $j_1 \le j_2 \le \ldots \le j_m < 0$ and if $j_i = j_{i-1} \Rightarrow \alpha_i \le \alpha_{i-1}$ w.r.t. given ordered set S.

Reconstruction theorem

Under assumptions above

$$Y(a^{\alpha}_{(j_1)} \ldots a^{\alpha}_{(j_m)}|0\rangle, z) = \frac{1}{(-j_1-1)! \ldots (-j_m-1)!} :\partial^{-j_1-1} a^{\alpha_1}(z) \ldots \partial^{-j_m-1} a^{\alpha_m}(z):$$

defines VOA structure on V, moreover if it is

$\mathbb{Z}$-graded

$|0\rangle$ – degree 0, a^α – homogeneous, T had deg 1

$\Rightarrow$ V – graded VA

Examples

1) Affine Kac-Moody Lie algebra

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$, Let's consider

$(\cdot\,,\cdot) = \frac{1}{2h^\vee}(\cdot\,,\cdot)_k$

$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0$

$[A \otimes f(t), B \otimes g(t)] = [A,B] \otimes f(t)g(t) - (\mathrm{Res}_{t=0} f\,dg)(A,B)_k$

$H^2(\mathcal{L}\mathfrak{g}, \mathbb{C})$ - 1-dim $\Rightarrow$ all central extensions are parametrized by $\mathbb{C}$

Vacuum representation:

$\mathfrak{g}[[t]] \subset \mathcal{L}\mathfrak{g}$
subalgebra of $\hat{\mathfrak{g}}$

$V_k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_k = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)} \mathbb{C}_k$

module induced from $\mathbb{C}_k$, where K acts by multiplication: $K\,\mathbb{C}_k = k\,\mathbb{C}_k$

PBW: $U(\hat{\mathfrak{g}}) = U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)$

As vector spaces: $V_k(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$

Basis $\{J^a\}$ $a = 1, \dots \dim \mathfrak{g}$

$$[J^a_n, J^b_m] = [J^a, J^b]_{m+n} + n(J^a, J^b)\delta_{n,-m} K$$

v_k – vector in $\mathbb{C}_k$

$J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_k$ $\quad n_1 \leq \dots \leq n_m < 0$ and if $n_i = n_{i+1}$ $\Rightarrow$ $a_i \leq a_{i+1}$

VOA structure:

$|0\rangle = v_k$, $T v_k = 0$, $[T, J^a_n] = -n J^a_{n-1}$

Vertex operators

$$Y(J^a_{-1} v_k, z) = J^a(z) = \sum J^a_n z^{-n-1}$$

$$Y(J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_k, z) = \frac{1}{(-n_1-1)! \dots (-n_m-1)!} :\partial_z^{-n_1-1} J^{a_1}(z) \dots \partial_z^{-n_m-1} J^{a_m}(z):$$

$\mathbb{Z}_+$-graded $\operatorname{Deg} J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_k = -\sum_{i=1}^{m} n_i$

<u>Theorem</u> $V_k(\mathfrak{g})$ is $\mathbb{Z}_+$-graded VOA

<u>Proof</u> $[J^a(z), J^b(w)] = [J^a, J^b](w)\,\delta(z-w) +$
$+ (J^a, J^b)\, k\, \partial_w \delta(z-w)$ $\Rightarrow$

$\Rightarrow (z-w)^2 [J^a(z), J^b(w)] = 0$ $\Rightarrow$ generators are local

Virasoro algebra

$\mathcal{K} = \mathbb{C}((t)) \quad \mathrm{Der}\,\mathcal{K} = \mathbb{C}((t))\partial_t$

$$0 \to \mathbb{C}c \to Vir \to \mathrm{Der}\,\mathcal{K} \to 0$$

$$[f(t)\partial_t, g(t)\partial_t] = (fg' - gf')\partial_t - \frac{c}{12}\left(\mathrm{Res}_{t=0} f g''' dt\right)$$

$$L_n = -t^{n+1}\partial_t \qquad n \in \mathbb{Z}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3-n)\delta_{n,-m}c$$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

Lemma

$$[T(z), T(w)] = \frac{c}{12}\partial_w^3\delta(z-w) + 2T(w)\partial_w\delta(z-w) + \partial_w T(w)\delta(z-w)$$

$$\mathrm{Der}\,\mathcal{O} = \mathrm{Der}\,\mathbb{C}[[t]]$$

Vacuum Module: $L_n|0\rangle = 0 \qquad n \geq -1$

$$Vir_c = \mathrm{Ind}^{Vir}_{\mathrm{Der}\,\mathcal{O}\oplus\mathbb{C}c}\mathbb{C}_c = U(Vir)\otimes_{U(\mathrm{Der}\,\mathcal{O}\oplus\mathbb{C}c)}\mathbb{C}_c$$

Construct VA (exercise)

Feigin-Fuks and Sugawara

Def. Conformal vertex algebra of central charge c is a $\mathbb{Z}$-graded vertex algebra with $\omega \in V_2$, s.t.
$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^{V} z^{-n-2}$ satisfy the rel. of Vir. algebra

and $L_{-1}^{V} = T$, $L_0^{V}|_{V_n} = n\,\mathrm{Id}$

Example π: $\omega_\lambda = \frac{1}{2} b_{-1}^2 + \lambda b_{-2}$ $\quad c_\lambda = 1 - 12\lambda^2$

$$Y(\omega_\lambda, z) = \frac{1}{2} :b(z)^2: + \lambda \partial_z b(z) = \sum \omega_{\lambda,n} z^{-n-2}$$

Exercise: check the relations

Example Segal-Sugawara const. for $V_k(\mathfrak{g})$

Let $k \neq -h^{\vee}$ $\quad S = \frac{1}{2} \sum_{a=1}^{d} J_{a,-1} J^{a}_{-1} v_k$ $\quad$ J^a - basis of $\mathfrak{g}$, J_a - dual basis w.r.t $(\cdot,\cdot)$

$$\frac{S}{k+h^{\vee}} \longrightarrow c(k) = \frac{k \dim(\mathfrak{g})}{k+h^{\vee}}$$

$$\frac{1}{k+h^{\vee}} Y(S,z) = \frac{1}{2(k+h^{\vee})} \sum_{a=1}^{d} :J_a(z) J^a(z):$$

Checking the comm. relation is rather difficult, so we will do it next time.