Lecture IX

Associativity vs Locality (Commutativity)

Forget about z-variable:

V, $Y: V \to \mathrm{End}(V)$ and $\mathbb{1} \in V$ s.t. $Y(B)\mathbb{1} = B$

with the property that $Y(A)Y(B) = Y(B)Y(A)$

Product structure: $Y(A) \cdot B = A \cdot B$.

(•) 1) commutative 2) associative

$$A \cdot (B \cdot C) = B \cdot (A \cdot C) \quad \forall C \in V$$

$$A \cdot (B \cdot C) = A \cdot (C \cdot B) = C \cdot (A \cdot B) = (A \cdot B) \cdot C$$

We will apply now this logic for VOAs

Some prerequisites

Goddard's uniqueness theorem

Let V be a VOA, $A(z)$ - field on V. Suppose $\exists a \in V$ s.t. $A(z) = Y(a,z)|0\rangle$ and $A(z)$ is local w.r.t $Y(b,z)$ $\forall b \in V \Rightarrow A(z) = Y(a,z)$

Proof. $(z-w)^N A(z) Y(b,w)|0\rangle = (z-w)^N Y(b,w) A(z)|0\rangle =$
$= (z-w)^N Y(b,w) Y(a,z)|0\rangle = (z-w)^N Y(a,z) Y(b,w)|0\rangle$

for N-large enough. Set $w=0$ $Y(a,z)b = A(z)b$ $\forall b \in V$

<u>Lemma</u> $Y(a,z)|0\rangle = e^{zT}a$

Proof. exercise $\quad a_{-n-1}|0\rangle = \frac{1}{n!}T^n a$

<u>Corollary</u> $Y(Ta,z) = \partial_z Y(a,z)$

Now: $Y(A,z)Y(B,w)C \in V((z))((w))$

$Y(B,w)Y(A,z)C \in V((w))((z))$

expansions of the same element from

$V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$

<u>Theorem</u> $\forall$ VOA V

$Y(A,z)Y(B,w)C \in V((z))((w))$

$Y(B,w)Y(A,z)C \in V((w))((z))$

$Y(Y(A,z-w)B,w)C \in V((w))((z-w))$

In analytic terms:

$$Y(Y(A,z-w)B,w) = \sum_{n\in\mathbb{Z}} Y(A_{(n)}B,w)(z-w)^{-n-1}$$

converges in the domain $|w| > |z-w|$ and can be continued to $R(Y(Y(A,z-w)B,w))$ - operator-valued meromorphic function with poles $z=0, w=0, z=w$

$$R(Y(A,z)Y(B,w)) = R(Y(Y(A,z-w)B),w))$$

<u>Lemma</u> $e^{wT}Y(A,z)e^{-Tw} = Y(A,z+w)$ in $\operatorname{End} V[[z^{\pm 1}, w^{\pm 1}]]$

where $(z+w)^{-k}$ we understand as expansions in $\mathbb{C}((z))((w))$, i.e. positive powers of $\frac{w}{z}$

Proof: $e^{wT}Y(A,z)e^{-Tw} = \sum_{n=0}^{\infty} \frac{w^n}{n!}\partial_z^n Y(A,z)$

<u>Proposition</u> (skew-symmetry)

$$Y(A,z)B = e^{zT}Y(B,-z)A \quad \text{in } V((z))$$

<u>Proof</u> $(z-w)^N Y(A,z)Y(B,w)|0\rangle =$

$= (z-w)^N Y(B,w)Y(A,z)|0\rangle$ for N large enough

$(z-w)^N Y(A,z)e^{wT}B = (z-w)^N Y(B,w)e^{zT}A$

$(z-w)^N Y(A,z)e^{wT}B = (z-w)^N e^{zT}Y(B,w-z)A$

$(w-z)^{-l}$ are understood as exp. in $\mathbb{C}((w))((z))$

Consider N large enough to kill all negative powers of $(w-z)$ $\Rightarrow$ then identify in $V((z))[[w]]$

then set $w=0$ and divide by z^N.

<u>Proof</u> $Y(A,z)Y(B,w)C = Y(A,z)e^{wT}Y(C,-w)B =$ in $\mathbb{C}[[z^{\pm1},w^{\pm1}]]$

$= e^{Tw}(e^{-Tw}Y(A,z)e^{wT})Y(C,-w)B$

Therefore $Y(A,z)Y(B,w)C = e^{wT}Y(A,z-w)Y(C,-w)B$ $(*)$

where by $(z-w)^{-1}$ we understand its expansion in positive powers in w/z

So, $Y(A,z)Y(B,w)C$ $\leftarrow$ expansions of the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$

$e^{wT}Y(A,z-w)Y(C,-w)B$ in $V((z))((w))$ and $V((z-w))((w))$

On the other hand,

$$Y(Y(A, z-w)B, w)\, C = \sum_{n\in\mathbb{Z}} (z-w)^{-n-1}\, Y(A_n B, w)\, C$$

$$Y(A_n \cdot B, w)\, C = e^{wT}\, Y(C, -w)\, A_n B \Rightarrow$$

$$\Rightarrow Y(Y(A, z-w)B, w)\, C = e^{wT}\, Y(C, -w)\, Y(A, z-w)\, B \quad (**)$$

Compare (**) and (*) we find that it is the expansion of the same element.

Operator product expansion

$Y(A, z)\, Y(B, w)\, C$. By locality it is the exp. in $V((z))((w))$ of $f_{ABC} \in V[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$

f_{ABC} in $V((w))((z-w))$ is $Y(Y(A, z-w)B, w)\, C$

the embedding is given by $z \to w + (z-w)$,

$z^{-1} \to (w + (z-w))^{-1}$ considered as $\frac{z-w}{w}$

$$Y(A, z)\, Y(B, w)\, C = Y(Y(A, z-w)B, w)\, C, \quad A, B, C \in V$$

or $$Y(A, z)\, Y(B, w)\, C = \sum_{n\in\mathbb{Z}} \frac{Y(A_n \cdot B, w)}{(z-w)^{n+1}}\, C$$

Operator product expansion

Proposition (Kac)

Let $\phi(z)$, $\psi(w)$ be arb. fields on V

then i), ii), iii) are equivalent:

i)

$$[\phi(z), \psi(w)] = \sum_{j=0}^{N-1} \frac{1}{j!} \gamma_j(w) \partial_w^j \delta(z-w)$$

where $\gamma_j(w)$ are some fields

ii) $\phi(z)\psi(w)$ (resp. $\psi(w)\phi(z)$)

$$= \sum_{j=0}^{N-1} \frac{\gamma_j(w)}{(z-w)^{j+1}} + :\phi(z)\psi(w): \text{ where}$$

$\frac{1}{z-w}$ is expanded in pos. powers of $\frac{w}{z}$ (resp. z/w)

iii) $\phi(z)\psi(w)$ conv. for $|z|>|w|$ to the expression above

and $\psi(w)\phi(z)$ conv. for $|w|>|z|$ to the same expr.

Proof: i) ⇒ ii) Remember: $f(z)_+ = \sum_{n\geq 0} f_n z^n$ $f(z)_- = \sum_{n<0} f_n z^n$

$[\phi(z)_\pm, \psi(w)] = (\sum \ldots)_\pm$ | $:\phi(z)\psi(w): =$
$= \psi(w)\phi(z)_- + \phi(z)_+\psi(w)$

$[\phi(z)_-, \psi(w)] = \phi(z)\psi(w) - :\phi(z)\psi(w):$

ii) ⇒ i) follows from $\delta_+(z-w) + \delta_-(z-w) = \delta(z-w)$

ii) ⇔ iii) is clear.

Normal ordering and OPE

Locality $(z-w)^N [Y(A,z), Y(B,w)] = 0 \Rightarrow$

$$\Rightarrow [Y(A,z), Y(B,w)] = \sum_{j=0}^{N-1} \frac{1}{j!} \gamma_j(w) \partial_w^j \delta(z-w)$$

$$Y(A,z)Y(B,w)C = \left(\sum_{j=0}^{N-1} \frac{\gamma_j(w)}{(z-w)^{j+1}} + :Y(A,z)Y(B,w): \right) C$$

Taylor expansion:

$$Y(A,z)Y(B,w)C = \left(\sum_{j=0}^{N-1} \frac{\gamma_j(w)}{(z-w)^{j+1}} + \sum_{m\geq 0} \frac{(z-w)^m}{m!} :\partial_w^m Y(A,w)Y(B,w): \right) C$$

in $V((w))((z-w))$

$$Y(A_{(n)} \cdot B, z) = \frac{1}{(-n-1)!} :\partial_z^{-n-1} Y(A,z) Y(B,z): \quad n < 0$$

since $Y(B,z)|0\rangle = e^{zT}B \Rightarrow B_{-n-1}|0\rangle = \frac{1}{n!} T^n B$

$$Y(B_{-n-1}|0\rangle, z) = \frac{1}{n!} \partial_z^n Y(B,z)$$

Corollary $\forall\ A^1, \dots, A^k \in V \quad n_1, \dots, n_k < 0$

$$Y(A^1_{n_1} \cdots A^k_{n_k} |0\rangle, z) = \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_k-1)!} :\partial_z^{-n_1-1} Y(A^1,z) \cdots \partial_z^{-n_k-1} Y(A^k,z):$$

Notice $\gamma_j(w) = Y(A_j \cdot B, w) \quad j \geq 0$

$$Y(A,z)Y(B,w) = \sum_{n\geq 0} \frac{Y(A_n \cdot B, w)}{(z-w)^{n+1}} + :Y(A,z)Y(B,w):$$

Unlike previous version, this makes sense in $\operatorname{End} V[[z^{\pm 1}, w^{\pm 1}]]$ so that $(z-w)^{-1}$ in $\mathbb{C}((z))((w))$

$$[Y(A,z), Y(B,w)] = \sum_{n\geq 0} \frac{1}{n!} Y(A_{(n)}\cdot B, w)\, \partial_w^n \delta(z-w)$$

$$[A_m, B_k] = \sum_{n\geq 0} \binom{m}{n} (A_n \cdot B)_{m+k-n}$$

Only the residue terms contribute to commutator!

$$Y(A,z)Y(B,w) \sim \sum_{n\geq 0} \frac{Y(A_n\cdot B, w)}{(z-w)^{-n-1}} \quad \text{(physics)}$$

<u>Corollary:</u> $[A_0, Y(B,w)] = Y(A_0\cdot B, w)$

<u>Examples:</u> $\oint T(z)\,dz = L_{-1}$ $\quad J_0^a = \int J^a(z)\,dz$