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QUANTUM INVERSE SCATTERING METHOD AND (SUPER)CONFORMAL FIELD THEORY

P. P. Kulish* **and A. M. Zeitlin†**

We consider the possibility of using the quantum inverse scattering method to study the superconformal field theory and its integrable perturbations. The classical limit of the considered constructions is based on the $\widehat{osp}(1|2)$ *super-KdV hierarchy. We introduce the quantum counterpart of the monodromy matrix corresponding to the linear problem associated with the L-operator and use the explicit form of the irreducible representations of* $\widehat{osp}_q(1|2)$ *to obtain the "fusion relations" for the transfer matrices (i.e., the traces of the monodromy matrices in different representations).*

Keywords: superconformal field theory, supersymmetric Korteweg–de Vries equation

1. Introduction

1.1. Quantum inverse scattering method. The quantum inverse scattering method (QISM) appeared in the works of the Leningrad school of mathematical physics in the late 1970s [1]. It arose as a synthesis of two approaches to integrable systems. The first approach, the so-called inverse scattering method (ISM), was discovered in 1967 [2] but has deep roots in the works on classical mechanics in the 19th century; the second approach was applied to problems in statistical physics on a lattice and in quantum mechanics [3] up to the end of the 1970s. The ISM has allowed finding classes (hierarchies) of integrable two-dimensional nonlinear evolution equations and obtaining their solutions. A few years after the ISM was discovered, the algebraic structure of this method was understood [4], and the Hamiltonian interpretation was obtained [5]. It turned out that the Hamiltonian systems corresponding to these equations are fully integrable and have infinitely many conservation laws. The algebraic structure of the ISM allows considering the integrable nonlinear equation under study as a compatibility condition of the system of linear equations

$$\partial_x\Psi = U(x,t,\lambda)\Psi, \tag{1}$$

$$\partial_t\Psi = V(x,t,\lambda)\Psi, \tag{2}$$

the so-called zero-curvature condition [6]

$$\partial_x V - \partial_t U + [V,U] = 0, \tag{3}$$

where the functions U and V take values in some Lie algebra $\mathbf{g}$. The Hamiltonian interpretation [5] allowed considering the transformation to the scattering data of linear problem (1) as a transformation to "action–angle" variables, in terms of which the problem reduces to a system of linear equations. This transformation and its inverse give the solution of the Cauchy problem. The corresponding family of the

*St. Petersburg Department of the Steklov Institute of Mathematics, RAS, St. Petersburg, Russia, e-mail: kulish@pdmi.ras.ru.
†St. Petersburg State University, St. Petersburg, Russia, e-mail: zam@math.ipme.ru.

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integrals of motion can be extracted from the spectral-parameter expansion of the trace of the monodromy matrix of Eq. (1). The Poisson brackets of elements of the monodromy matrix $T(\lambda)$ for different values of the spectral parameter often have the form

$$\{T(\lambda)\otimes, T(\mu)\} = [r(\lambda\mu^{-1}), T(\lambda)\otimes T(\mu)], \tag{4}$$

where r is a classical r-matrix [6]. With this relation, it can be easily shown that $\{t(\lambda), t(\mu)\} = 0$, where $t(\lambda) = \operatorname{tr} T(\lambda)$, which is the integrability condition ensuring the existence of an involutive family of integrals of motion. After quantization, (4) transforms into the RTT relation

$$R(\lambda\mu^{-1})(T^{(q)}(\lambda)\otimes I)(I\otimes T^{(q)}(\mu)) = (I\otimes T^{(q)}(\mu))(T^{(q)}(\lambda)\otimes I)R(\lambda\mu^{-1}), \tag{5}$$

where R is a quantum R-matrix [7], [8]. It can be easily shown that $[t^{(q)}(\lambda), t^{(q)}(\mu)] = 0$, as was the case at the classical level; in other words, we obtain quantum integrability, the existence of pairwise commuting operators. Relation (5) is the starting point of the already mentioned second approach to the theory of integrable systems. Using the RTT relation with different methods, for example, the algebraic Bethe ansatz, we can find the spectrum of the transfer matrix $t^{(q)}(\lambda)$ and different correlators. To obtain (5) for two-dimensional field theory systems, it is often necessary to consider them on a lattice. But for some systems, such as the KdV equation, it is possible to construct the RTT relation and find the explicit form of the monodromy matrix using the continuum field theory [9]–[11]. The KdV equation also allows quantization in another way, involving the boson–fermion correspondence [12].

In this paper, we extend this class of systems by including a supersymmetric generalization of the KdV equation [13] and show the peculiarities appearing in the quantization in terms of a continuum field theory of supersymmetric KdV hierarchies. A preliminary version of this paper has been published [14].

1.2. (Super)conformal field theory, its perturbations, and the QISM. In 1970, it was hypothesized [15] that the field theory corresponding to a fixed (critical) point of the renormalization group has not only scale but also conformal invariance. In two dimensions, because $2d$ conformal symmetry is infinite-dimensional and is related to the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}, \tag{6}$$

it is possible to classify all the fields in the theory and calculate correlation functions.

Perturbations usually break the conformal symmetry and take the system out of the critical point. But perturbations of the special type called “integrable” [16] nevertheless preserve the infinite involutive algebra of the integrals of motion and thus lead to an integrable theory. In [9]–[11], it was shown that in this case, the problem can be solved using the QISM in terms of continuous fields. The following was proposed: first use the conformal symmetry to build basic structures of the QISM at the critical point and then study the perturbed theory using the QISM. Our object of study is a model based on a supersymmetric generalization of conformal symmetry (superconformal symmetry) [17], [18]. We use its properties to build the quantum monodromy matrix, the RTT relation, and fusion rules for the transfer matrices in different representations.

As a classical limit of this quantum model, we consider the theory of the $\widehat{osp}(1|2)$-supersymmetric KdV (super-KdV) equation [13]. We introduce two equivalent L-operators, the Miura transform and the Poisson brackets corresponding to the Drinfeld–Sokolov reduction of the affine superalgebra $\widehat{osp}(1|2)$, and also construct the associated monodromy matrix. Its supertrace is the generating function for both local and nonlocal integrals of motion that are in involution with respect to the Poisson brackets. Using the monodromy matrix, we then introduce the auxiliary matrix $\mathbf{L}(\lambda)$, whose Poisson brackets have form (4) for different values of the spectral parameter.

After these necessary preparations, we proceed to the quantum theory (Sec. 3). We give a quantum version of the Miura transformation, a free-field representation of the superconformal algebra [17], [18], and introduce the vertex operators necessary for constructing the quantum monodromy matrix.

In the quantum case, the algebraic structure of the monodromy matrix is described in terms of the affine superalgebra $\widehat{osp}_q(1|2)$. Its representations are constructed in Sec. 4. We then introduce the quantum counterpart of the auxiliary $\mathbf{L}$-matrix. It turns out that one term is absent from the P-exponential in the quantum case, while it is present in both the monodromy matrix and the auxiliary matrix in the classical case. Quantum $\mathbf{L}$-matrices satisfy the RTT relation, thus providing the integrability in the quantum case. Considering the monodromy matrix in different representations of $\widehat{osp}_q(1|2)$, we obtain the functional relations (fusion relations) for their supertraces, the "transfer matrices." In the cases where the deformation parameter is rational, which correspond to the minimal models of (super)conformal field theory, the fusion relations become a closed system of equations, which, following the conjecture in [9], can be used to find the complete set of the eigenvalues of transfer matrices. Moreover, we believe that the equations in that system can be transformed into the thermodynamic Bethe ansatz equations [19].

2. A review of the classical super-KdV theory

The quantization of the Drinfeld–Sokolov hierarchies of the KdV type related to the affine algebras $A_1^{(1)}$, $A_2^{(2)}$, and $A_2^{(1)}$ was given in [9]–[11]. In the classical limit, our quantum model gives the super-KdV hierarchy [13] related to the affine superalgebra $\widehat{osp}(1|2)$. The supermatrix L-operator corresponding to the super-KdV theory is given by

$$\mathcal{L}_{\mathrm{F}} = D_{u,\theta} - D_{u,\theta}\Psi h - (iv_+\sqrt{\lambda} - \theta\lambda X_-), \tag{7}$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative, the variable u lies on a cylinder of circumference 2π, θ is a Grassmann variable, $\Psi(u,\theta) = \phi(u) - i\theta\xi(u)/\sqrt{2}$ is a bosonic superfield, and h, v_+, v_-, X_-, and X_+ are generators of $osp(1|2)$ (see [20] for more information):

$$\begin{aligned} &[h, X_\pm] = \pm 2X_\pm, \qquad [h, v_\pm] = \pm v_\pm, \qquad [X_+, X_-] = h, \\ &[v_\pm, v_\pm] = \pm 2X_\pm, \qquad [v_+, v_-] = -h, \qquad [X_\pm, v_\mp] = v_\pm, \qquad [X_\pm, v_\pm] = 0. \end{aligned} \tag{8}$$

Here, $[\,\cdot\,,\cdot\,]$ denotes the supercommutator, $[a,b] = ab - (-1)^{p(a)p(b)}ba$, and the parity p is defined as $p(v_\pm) = 1$, $p(X_\pm) = 0$, $p(h) = 0$. The "fermionic" operator $\mathcal{L}_{\mathrm{F}}$ considered together with the linear problem $\mathcal{L}_{\mathrm{F}}\chi(u,\theta) = 0$ is equivalent to the "bosonic" one:

$$\mathcal{L}_{\mathrm{B}} = \partial_u - \phi'(u)h - \sqrt{\frac{\lambda}{2}}\xi(u)v_+ - \lambda(X_+ + X_-). \tag{9}$$

The fields ϕ and ξ satisfy the boundary conditions

$$\phi(u+2\pi) = \phi(u) + 2\pi i p, \qquad \xi(u+2\pi) = \pm\xi(u), \tag{10}$$

where the plus and minus signs respectively correspond to the so-called Ramond (R) and Neveu–Schwarz (NS) sectors of the model. The Poisson brackets given by the Drinfeld–Sokolov construction are

$$\{\xi(u), \xi(v)\} = -2\delta(u-v), \qquad \{\phi(u), \phi(v)\} = \frac{1}{2}\epsilon(u-v). \tag{11}$$

Written in the Miura form, L-operators (7) and (9) correspond to the super-mKdV hierarchy. Making a gauge transformation to proceed to the super-KdV L-operator, we obtain the two fields

$$\begin{aligned} U(u) &= -\phi''(u) - \phi'^2(u) - \frac{1}{2}\xi(u)\xi'(u), \\ \alpha(u) &= \xi'(u) + \xi(u)\phi'(u), \end{aligned} \tag{12}$$

which generate the superconformal algebra under the Poisson brackets:

$$\begin{aligned} \{U(u), U(v)\} &= \delta'''(u-v) + 2U'(u)\delta(u-v) + 4U(u)\delta'(u-v), \\ \{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u-v) + \alpha'(u)\delta(u-v), \\ \{\alpha(u), \alpha(v)\} &= 2\delta''(u-v) + 2U(u)\delta(u-v). \end{aligned} \tag{13}$$

These brackets describe the second Hamiltonian structure of the super-KdV hierarchy. We can obtain an evolution equation by taking one of the corresponding infinite set of local integrals of motion

$$\begin{aligned} I_1^{(\mathrm{cl})} &= \int U(u)\,du, \\ I_3^{(\mathrm{cl})} &= \int \left(\frac{U^2(u)}{2} + \alpha(u)\alpha'(u)\right) du, \\ I_5^{(\mathrm{cl})} &= \int \big((U')^2(u) - 2U^3(u) + 8\alpha'(u)\alpha''(u) + 12\alpha'(u)\alpha(u)U(u)\big)\,du, \\ &\ \ \vdots \end{aligned} \tag{14}$$

(they can be obtained from the λ-expansion of the supertrace of the monodromy matrix; see below). These integrals of motion form an involutive set with respect to the Poisson brackets,

$$\{I_{2k-1}^{(\mathrm{cl})}, I_{2l-1}^{(\mathrm{cl})}\} = 0,$$

and $I_3^{(\mathrm{cl})}$ leads to the super-KdV equation [13]

$$\begin{aligned} U_t &= -U_{uuu} - 6UU_u - 6\alpha\alpha_{uu}, \\ \alpha_t &= -4\alpha_{uuu} - 6U\alpha_u - 3U_u\alpha. \end{aligned} \tag{15}$$

We now consider the "bosonic" linear problem $\pi_s(\mathcal{L}_{\mathrm{B}})\chi(u) = 0$, where π_s denotes the irreducible representation of $osp(1|2)$ labeled by an integer $s \geq 0$ [20]. We can write the solution of this problem as

$$\chi(u) = \pi_s\Big(e^{\phi(u)h}\,\mathrm{Pexp}\int_0^u du'\,\Big(\sqrt{\frac{\lambda}{2}}\xi(u')v_+e^{-\phi(u')} + \lambda(X_+e^{-2\phi(u')} + X_-e^{2\phi(u')})\Big)\Big)\chi_0,$$

where Pexp denotes the P-ordered exponential and $\chi_0 \in \mathbb{C}^{2s+1}$ is a constant vector. This can be rewritten in a more general form as

$$\chi(u) = \pi_s(\lambda)\Big(e^{-\phi(u)h_{\alpha_0}}\,\mathrm{Pexp}\int_0^u du'\,\big(\xi(u')e^{-\phi(u')}e_\alpha + e^{-2\phi(u')}2e_\alpha^2 + e^{2\phi(u')}e_{\alpha_0}\big)\Big)\chi_0,$$

where e_α, e_{α_0}, and h_{α_0} are those of the Chevalley generators of $\widehat{osp}(1|2)$ (see [21]) that respectively coincide with $\sqrt{\lambda/2}v_+$, λX_-, and $-h$ in the evaluation representations $\pi_s(\lambda)$. The associated monodromy matrix then has the form

$$\mathbf{M}_s(\lambda) = \pi_s\biggl(e^{-2\pi i p h_{\alpha_0}}\,\mathrm{Pexp}\int_0^{2\pi} du'\,\bigl(\xi(u')e^{-\phi(u')}e_\alpha + e^{-2\phi(u')}2e_\alpha^2 + e^{2\phi(u')}e_{\alpha_0}\bigr)\biggr).$$

Following [9]–[11], we introduce the auxiliary matrices

$$\pi_s(\lambda)(\mathbf{L}) = \mathbf{L}_s(\lambda) = \pi_s(\lambda)(e^{\pi i p h_{\alpha_0}})\mathbf{M}_s(\lambda). \tag{16}$$

They satisfy Poisson-bracket algebra (4),

$$\{\mathbf{L}_s(\lambda) \otimes, \mathbf{L}_{s'}(\mu)\} = [\mathbf{r}_{ss'}(\lambda\mu^{-1}), \mathbf{L}_s(\lambda) \otimes \mathbf{L}_{s'}(\mu)], \tag{17}$$

where $\mathbf{r}_{ss'}(\lambda\mu^{-1}) = \pi_s(\lambda) \otimes \pi_{s'}(\mu)(\mathbf{r})$ is the classical trigonometric $\widehat{osp}(1|2)$ r-matrix [22] taken in the corresponding representations:

$$\begin{aligned}\mathbf{r}(\lambda\mu^{-1}) = \frac{1}{2}\frac{\lambda\mu^{-1}+\lambda^{-1}\mu}{\lambda\mu^{-1}-\lambda^{-1}\mu}h\otimes h + \frac{2}{\lambda\mu^{-1}-\lambda^{-1}\mu}(X_+\otimes X_- + X_-\otimes X_+) +{}\\ + \frac{1}{(\lambda\mu^{-1}-\lambda^{-1}\mu)}\biggl(\sqrt{\frac{\mu}{\lambda}}v_+\otimes v_- - \sqrt{\frac{\lambda}{\mu}}v_-\otimes v_+\biggr).\end{aligned} \tag{18}$$

From the Poisson brackets for $\mathbf{L}_s(\lambda)$, we find that the traces of the monodromy matrices commute with respect to the Poisson brackets: $\{\mathbf{t}_s(\lambda), \mathbf{t}_{s'}(\mu)\} = 0$. Expanding $\log(\mathbf{t}_1(\lambda))$ in a series in λ^{-1}, we can see that the coefficients in this expansion are local integrals of motion, as noted above.

3. Free-field representation of the superconformal algebra and vertex operators

To introduce quantum analogues of classical objects such as the monodromy matrix, we start from a quantum version of the Miura transformation (12), the so-called free-field representation of the superconformal algebra [17],

$$\begin{aligned}-\beta^2 T(u) &= {:}\phi'^{\,2}(u){:} + \biggl(1-\frac{\beta^2}{2}\biggr)\phi''(u) + \frac{1}{2}{:}\xi\xi'(u){:} + \frac{\epsilon\beta^2}{16},\\ \frac{i^{1/2}\beta^2}{\sqrt{2}}G(u) &= \phi'\xi(u) + \biggl(1-\frac{\beta^2}{2}\biggr)\xi'(u),\end{aligned} \tag{19}$$

where

$$\begin{aligned}&\phi(u) = iQ + iPu + \sum_n \frac{a_{-n}}{n}e^{inu}, \qquad \xi(u) = i^{-1/2}\sum_n \xi_n e^{-inu},\\ &[Q,P] = \frac{i}{2}\beta^2, \qquad [a_n, a_m] = \frac{\beta^2}{2}n\delta_{n+m,0}, \qquad \{\xi_n, \xi_m\} = \beta^2\delta_{n+m,0}.\end{aligned} \tag{20}$$

The parameter β^2 plays the role of a semiclassical parameter (Planck's constant). We recall that there are two types of boundary conditions for ξ: $\xi(u+2\pi) = \pm\xi(u)$. The plus sign corresponds to the R sector, the case where ξ contains only integer modes; the minus sign corresponds to the NS sector, the case where ξ contains only half-integer modes. The variable ϵ in (19) is equal to zero in the R case and to one in the NS case.

We can expand $T(u)$ and $G(u)$ in modes as

$$T(u) = \sum_n L_{-n} e^{inu} - \frac{\hat{c}}{16}, \qquad G(u) = \sum_n G_{-n} e^{inu}, \tag{21}$$

where the central charge is $\hat{c} = 5 - 2(\beta^2/2 + 2/\beta^2)$ and L_n and G_m generate the superconformal algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n,-m}, \\
[L_n, G_m] &= \left(\frac{n}{2} - m\right)G_{m+n}, \\
[G_n, G_m] &= 2L_{n+m} + \delta_{n,-m}\frac{\hat{c}}{2}\left(n^2 - \frac{1}{4}\right).
\end{aligned} \tag{22}$$

In the classical limit $\hat{c} \to -\infty$ (or, equivalently, $\beta^2 \to 0$), the substitution

$$T(u) \to -\frac{\hat{c}}{4}U(u), \qquad G(u) \to -\frac{\hat{c}}{2\sqrt{2i}}\alpha(u), \qquad [\,\cdot\,,\cdot\,] \to \frac{4\pi}{i\hat{c}}\{\,\cdot\,,\cdot\,\}$$

reduces the above algebra to the Poisson-bracket algebra of the super-KdV theory.

Let F_p be the Fock representation with the vacuum (highest-weight) vector $|p\rangle$. The vector $|p\rangle$ is determined by the eigenvalue of P and the annihilation condition for positive-mode generators:

$$P|p\rangle = p|p\rangle, \qquad a_n|p\rangle = 0, \qquad \xi_m|p\rangle = 0, \quad n, m > 0. \tag{23}$$

In the R sector, the highest weight becomes doubly degenerate because the zero mode ξ_0 is present, i.e., there are two ground states $|p,+\rangle$ and $|p,-\rangle$ such that $|p,+\rangle = \xi_0|p,-\rangle$. Using free-field representation (19) of the superconformal algebra, we can find that for general $\hat{c}$ and p, F_p is isomorphic to the super-Virasoro module with the highest-weight vector $|p\rangle$ in the NS sector,

$$L_0|p\rangle = \Delta_{\mathrm{NS}}|p\rangle, \qquad \Delta_{\mathrm{NS}} = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}-1}{16}, \tag{24}$$

and to the module with two highest-weight vectors in the R sector,

$$\begin{aligned}
L_0|p,\pm\rangle &= \Delta_{\mathrm{R}}|p,\pm\rangle, \qquad \Delta_{\mathrm{R}} = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}}{16}, \\
|p,+\rangle &= \frac{\beta^2}{\sqrt{2}p}G_0|p,-\rangle.
\end{aligned} \tag{25}$$

Considered as a super-Virasoro module, the space F_p decomposes into the sum of finite-dimensional subspaces determined by the value of L_0:

$$F_p = \bigoplus_{k=0}^{\infty} F_p^{(k)}, \qquad L_0 F_p^{(k)} = (\Delta + k) F_p^{(k)}. \tag{26}$$

The quantum versions of local integrals of motion should act invariantly on the subspaces $F_p^{(k)}$. Therefore, the problem of diagonalizing integrals of motion in the infinite subspace F_p reduces to a finite problem in each subspace $F_p^{(k)}$, but this problem rapidly becomes very complex for large k. We also note that in the R sector, G_0 does not commute with integrals of motion (even classically), and integrals of motion therefore mix $|p,+\rangle$ and $|p,-\rangle$.

Finally, we introduce another useful notion, the vertex operator. We need two types of them, "bosonic" and "fermionic,"

$$V_{\mathrm{B}}^{(a)} = \int d\theta\, \theta {:}e^{a\Psi}{:}, \qquad V_{\mathrm{F}}^{(b)} = \int d\theta\, {:}e^{b\Psi}{:}, \tag{27}$$

where $\Psi(u,\theta) = \phi(u) - i\theta\xi(u)/\sqrt{2}$ is a superfield, and hence

$$V_{\mathrm{B}}^{(a)} = {:}e^{a\phi}{:}, \qquad V_{\mathrm{F}}^{(b)} = -\frac{ib}{\sqrt{2}} \xi {:}e^{b\phi}{:}, \tag{28}$$

where normal ordering means that

$${:}e^{c\phi(u)}{:} = \exp\biggl(c\sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{inu}\biggr) \exp(ci(Q + Pu)) \exp\biggl(-c\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-inu}\biggr). \tag{29}$$

4. Quantum monodromy matrix and fusion relations

In this section, we construct quantum versions of monodromy matrices, the operators $\mathbf{L}_s$ and $\mathbf{t}_s$. The classical monodromy matrix is based on the $\widehat{osp}(1|2)$ affine Lie algebra. In the quantum case, the underlying algebra is the quantum $\widehat{osp}_q(1|2)$ [21] with $q = e^{i\pi\beta^2}$ and the generators corresponding to the even root α_0 and the odd root α:

$$\begin{aligned}
&[h_\gamma, h_{\gamma'}] = 0 \quad (\gamma, \gamma' = \alpha, d, \alpha_0), \\
&[e_\beta, e_{\beta'}] = \delta_{\beta,-\beta'}[h_\beta] \quad (\beta = \alpha, \alpha_0), \\
&[h_d, e_{\pm\alpha_0}] = \pm e_{\pm\alpha_0}, \qquad [h_d, e_{\pm\alpha}] = 0, \\
&[h_{\alpha_0}, e_{\pm\alpha_0}] = \pm 2e_{\pm\alpha_0}, \qquad [h_{\alpha_0}, e_{\pm\alpha}] = \mp e_{\pm\alpha}, \\
&[h_\alpha, e_{\pm\alpha}] = \pm\frac{1}{2}e_{\pm\alpha}, \qquad [h_\alpha, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0}, \\
&[[e_{\pm\alpha}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0, \\
&[e_{\pm\alpha}, [e_{\pm\alpha}, [e_{\pm\alpha}[e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm\alpha_0}]_q]_q]_q]_q]_q = 0.
\end{aligned} \tag{30}$$

Here, $[\,\cdot\,,\,\cdot\,]_q$ is the super q-commutator, $[e_a, e_b]_q = e_a e_b - q^{(a,b)}(-1)^{p(a)p(b)} e_b e_a$, with the parity p defined as $p(h_{\alpha_0}) = 0$, $p(h_\alpha) = 0$, $p(e_{\pm\alpha_0}) = 0$, $p(e_{\pm\alpha}) = 1$. Also, as usual, $[h_\beta] = (q^{h_\beta} - q^{-h_\beta})/(q - q^{-1})$. The finite-dimensional $\widehat{osp}_q(1|2)$-representations $\pi_s^{(q)}(\lambda)$ can be characterized by an integer number s and are explicitly given by

$$\begin{aligned}
&h_{\alpha_0}|j,m\rangle = 2m|j,m\rangle, \qquad e_{\alpha_0}|j,m\rangle = \lambda\sqrt{[j-m][j+m+1]}|j,m+1\rangle,\\
&e_{-\alpha_0}|j,m\rangle = \lambda^{-1}\sqrt{[j+m][j-m+1]}|j,m-1\rangle,\\
&e_{\alpha}|j,m\rangle = \sqrt{\lambda}\Big((-1)^{-2j}\sqrt{\alpha(j)[j-m+1]}|j+1/2,m-1/2\rangle +\\
&\qquad + \sqrt{\alpha(j-1/2)[j+m]}|j-1/2,m-1/2\rangle\Big),\\
&e_{-\alpha}|j,m\rangle = (\sqrt{\lambda})^{-1}\Big(-\sqrt{\alpha(j)[j+m+1]}|j+1/2,m+1/2\rangle -\\
&\qquad - (-1)^{2j}\sqrt{\alpha(j-1/2)[j-m]}|j-1/2,m+1/2\rangle\Big),\\
&h_{\alpha_0} = -2h_{\alpha}, \qquad h_d = \frac{1}{2}\lambda\frac{d}{d\lambda} + \frac{1}{4}h_{\alpha_0},
\end{aligned} \tag{31}$$

where $j = 0, 1/2, \ldots, s/2$ and $m = -j, -j+1, \ldots, j$. The normalization coefficients

$$\alpha(j) = \frac{[j+1][j+1/2][1/4]}{[2j+2][2j+1][1/2]}\left((-1)^{s-2j+1}\frac{[s+3/2]}{[s/2+3/4]} + \frac{[j+3/2]}{[j/2+3/4]}\right)$$

are defined by the recursive relation

$$\alpha(j)\frac{[2j+2]}{[j+1]} + \alpha\left(j-\frac{1}{2}\right)\frac{[2j]}{[j]} = 1, \qquad \alpha\left(\frac{s}{2}\right) = 0. \tag{32}$$

It is easy to see that in the classical limit $q \to 1$, $\alpha(s/2-k) = 0$ if $k < s/2$ is a nonnegative integer and $\alpha(s/2-k) = 1/2$ if $k < s/2$ is a nonnegative half-integer. Using this fact, we can find that in the classical limit, this representation becomes a direct sum of finite-dimensional irreducible representations of $\widehat{osp}(1|2)$:

$$\pi_s^{(1)}(\lambda) = \bigoplus_{k=0}^{[s/2]} \pi_{s-2k}(\lambda). \tag{33}$$

In this sum, k ranges integer numbers. We note that the structure of irreducible finite-dimensional representations of $\widehat{osp}_q(1|2)$ is similar to that of those of $(A_2^{(2)})_q$ [11]. This is a consequence of the coincidence of their Cartan matrices.

After these preparations, we are ready to introduce the quantum counterparts of the $\mathbf{L}_s$ operators:

$$\mathbf{L}_s^{(q)} = \pi_s^{(q)}(\lambda)(\mathbf{L}^{(q)}) = \pi_s^{(q)}\left(e^{-i\pi P h_{\alpha_0}}\,\mathrm{Pexp}\left(\int_0^{2\pi}\left(:e^{2\phi(u)}:e_{\alpha_0} + \xi(u):e^{-\phi(u)}:e_{\alpha}\right)du\right)\right).$$

We see that one term is missing in the P-exponential in comparison with the classical case (16). The terms corresponding to composite roots also disappear for general super-KdV hierarchies (we will return to this elsewhere).

Analyzing the singularity properties of the integrands in the P-exponential of $\mathbf{L}_s^{(q)}(\lambda)$, we can find that the integrals converge for $\hat{c}$ in the interval $(-\infty, 0)$. Using regularization, we can continue the P-exponential to a wider region of $\hat{c}$.

We now prove that the $\mathbf{L}^{(q)}$ coincide with $\mathbf{L}$ in the classical limit. We first analyze the products of the operators involved in the P-exponential. The product of two fermion operators can be written as

$$\xi(u)\xi(u') = :\xi(u)\xi(u'): - i\beta^2 \frac{e^{-\kappa i(u-u')/2}}{e^{i(u-u')/2} - e^{-i(u-u')/2}}, \tag{34}$$

where $\kappa = 0$ in the NS sector and $\kappa = 1$ in the R sector. For vertex operators, the corresponding operator product is

$$:e^{a\phi(u)}::e^{b\phi(u')}: = \left(e^{i(u-u')/2} - e^{-i(u-u')/2}\right)^{ab\beta^2/2} :e^{a\phi(u)+b\phi(u')}:, \tag{35}$$

where

$$:e^{a\phi(u)+b\phi(u')}: = \exp\left(a\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu} + b\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu'}\right)\times$$
$$\times \exp\bigl(ai(Q+Pu) + bi(Q+Pu')\bigr)\exp\left(-a\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu} - b\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu'}\right). \tag{36}$$

It is useful to rewrite these products with the singular part isolated,

$$\xi(u)\xi(u') = -\frac{i\beta^2}{iu - iu'} + \sum_{k=1}^{\infty} c_k(u)(iu - iu')^k, \tag{37}$$

$$:e^{a\phi(u)}::e^{b\phi(u')}: = (iu - iu')^{ab\beta^2/2}\left(:e^{(a+b)\phi(u)}: + \sum_{k=1}^{\infty} d_k(u)(iu - iu')^k\right), \tag{38}$$

where $c_k(u)$ and $d_k(u)$ are operator-valued functions of u. We now return to the $\mathbf{L}^{(q)}(\lambda)$ operator, which we can express as

$$\mathbf{L}^{(q)} = e^{-i\pi P h_{\alpha_0}} \lim_{N\to\infty}\prod_{m=1}^{N}\tau_m^{(q)}, \tag{39}$$

where

$$\tau_m^{(q)} = \mathrm{Pexp}\int_{x_{m-1}}^{x_m} K(u)\,du, \qquad K(u) \equiv :e^{2\phi(u)}:e_{\alpha_0} + \xi(u):e^{-\phi(u)}:e_\alpha.$$

Here, we divide the interval $[0, 2\pi]$ into intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \Delta = 2\pi/N$. We now consider the behavior of the first two iterations as $\beta^2 \to 0$:

$$\tau_m^{(q)} = 1 + \int_{x_{m-1}}^{x_m} K(u)\,du + \int_{x_{m-1}}^{x_m} K(u)\,du \int_{x_{m-1}}^{u} K(u')\,du' + O(\Delta^2). \tag{40}$$

It turns out that in the limit as $\beta^2 \to 0$, the second-iteration terms can contribute to the first iteration. To see this, we consider the expression that results from the second iteration:

$$-\int_{x_{m-1}}^{x_m} du\,\xi(u)\int_{x_{m-1}}^{u} du'\,\xi(u'):e^{-\phi(u)}::e^{-\phi(u')}:e_\alpha^2. \tag{41}$$

Using the above operator products and seeking the terms of the order $\Delta^{1+\beta^2}$ (only these can give the first-iteration terms in the $\beta^2 \to 0$ limit), we find that their contribution is

$$i\beta^2\int_{x_{m-1}}^{x_m} du\int_{x_{m-1}}^{u} du'\,(iu - iu')^{\beta^2/2-1}:e^{-2\phi(u)}:e_\alpha^2 = 2\int_{x_{m-1}}^{x_m} du:e^{-2\phi(u)}:(iu - ix_{m-1})^{\beta^2/2}e_\alpha^2. \tag{42}$$

Considering this in the classical limit, we recognize the familiar terms from $\mathbf{L}$:

$$\tau_m^{(1)} = 1 + \int_{x_{m-1}}^{x_m} du\,\big(\xi(u)e^{-\phi(u)}e_\alpha + e^{2\phi(u)}e_{\alpha_0} + e^{-2\phi(u)}2e_\alpha^2\big) + O(\Delta^2). \tag{43}$$

Collecting all $\tau_m^{(1)}$, we obtain the desired result

$$\mathbf{L}^{(1)} = \mathbf{L}. \tag{44}$$

Recalling the structure of the $\widehat{osp}_q(1|2)$ representations, we obtain

$$\mathbf{L}_s^{(1)}(\lambda) = \sum_{k=0}^{[s/2]} \mathbf{L}_{s-2k}(\lambda). \tag{45}$$

It follows from the properties of the quantum R-matrix [7] that $\mathbf{R}\Delta(\mathbf{L}^{(q)}) = \Delta^{\mathrm{op}}(\mathbf{L}^{(q)})\mathbf{R}$, where Δ and Δ^{op} are the respective coproduct and opposite coproduct of $\widehat{osp}_q(1|2)$ [21]. Factoring $\Delta(\mathbf{L}^{(q)})$ and $\Delta^{\mathrm{op}}(\mathbf{L}^{(q)})$ according to the properties of vertex operators and the P-exponential, we obtain the so-called RTT relation [7], [8]

$$\mathbf{R}_{ss'}(\lambda\mu^{-1})(\mathbf{L}_s^{(q)}(\lambda)\otimes\mathbf{I})(\mathbf{I}\otimes\mathbf{L}_{s'}^{(q)}(\mu)) = (\mathbf{I}\otimes\mathbf{L}_{s'}^{(q)}(\mu))(\mathbf{L}_s^{(q)}(\lambda)\otimes\mathbf{I})\mathbf{R}_{ss'}(\lambda\mu^{-1}), \tag{46}$$

where $\mathbf{R}_{ss'}$ is the trigonometric solution of the corresponding Yang–Baxter equation [22], which acts in the space $\pi_s(\lambda)\otimes\pi_{s'}(\mu)$.

We now define the "transfer matrices," which are quantum counterparts of the traces of monodromy matrices:

$$\mathbf{t}_s^{(q)}(\lambda) = \operatorname{str}\pi_s(\lambda)(e^{-i\pi P h_{\alpha_0}}\mathbf{L}_s^{(q)}). \tag{47}$$

According to the RTT relation, we obtain

$$\big[\mathbf{t}_s^{(q)}(\lambda), \mathbf{t}_{s'}^{(q)}(\mu)\big] = 0. \tag{48}$$

For the first nontrivial representation ($s = 1$), it is quite easy to find the expression for $\mathbf{t}_1^{(q)}(\lambda) \equiv \mathbf{t}^{(q)}(\lambda)$,

$$\mathbf{t}^{(q)}(\lambda) = 1 - 2\cos(2\pi i P) + \sum_{n=1}^{\infty}\lambda^{2n}Q_n, \tag{49}$$

where Q_n are nonlocal integrals of motion.

As in (48), $\mathbf{t}^{(q)}(\lambda)$ is the generating function for pairwise commuting nonlocal conservation laws: $[Q_n, Q_m] = 0$. We also suppose that $\mathbf{t}^{(q)}(\lambda)$ generates local integrals of motion in the classical case. Using (48) again and expanding $\log(\mathbf{t}^{(q)}(\lambda))$, we obtain

$$\big[Q_n, I_{2k-1}^{(q)}\big] = 0, \qquad \big[I_{2l-1}^{(q)}, I_{2k-1}^{(q)}\big] = 0. \tag{50}$$

The first few orders of the expansion of $\mathbf{t}_s^{(q)}(\lambda)$ operators in λ^2 yield the fusion relation

$$\mathbf{t}_s^{(q)}(q^{1/4}\lambda)\mathbf{t}_s^{(q)}(q^{-1/4}\lambda) = \mathbf{t}_{s+1}^{(q)}\big(q^{1/(2\beta^2)}\lambda\big)\mathbf{t}_{s-1}^{(q)}\big(q^{1/(2\beta^2)}\lambda\big) + \mathbf{t}_s^{(q)}(\lambda). \tag{51}$$

This result resembles the fusion relation for $(A_2^{(2)})_q$ [11]. This correspondence should not seem extraordinary because the representations of the corresponding algebras are very similar.

Note added in proof. New results concerning the quantization of another (super) generalization of the KdV equation appeared in [23]–[25].

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