Field Equations, Homotopy Gerstenhaber Algebras and Courant Algebroids

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Outline

Sigma-models and conformal invariance conditions

Sigma-models for string theory in curved spacetimes:

Let $X : \Sigma \to M$, where Σ is a compact Riemann surface (worldsheet) and M is a Riemannian manifold (target space).

Action functional of sigma model:

$$S_{so} = \frac{1}{4\pi h}\int_\Sigma (G_{\mu\nu}(X)dX^\mu \wedge * dX^\nu + X^*B)$$

where G is a metric on M, B is a 2-form on M.

Symmetries:

i) conformal symmetry on the worldsheet,

ii) diffeomorphism symmetry and $B \to B + d\lambda$ on target space.

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$$S_{so} \to S_{so}^{\Phi} = S_{so} + \int_{\Sigma} \Phi(X) R^{(2)}(\gamma) \mathrm{vol}_{\Sigma},$$

where function Φ is called *dilaton*, γ is a metric on Σ.

In order to make sense of path integral

$$Z = \int DX \quad e^{-S_{so}^{\Phi}(X,\gamma)}$$

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Conformal invariance conditions

$$\mu\frac{d}{d\mu}G_{\mu\nu}=\beta^G_{\mu\nu}(G,B,\Phi,h)=0,\quad \mu\frac{d}{d\mu}B_{\mu\nu}=\beta^B_{\mu\nu}(G,B,\Phi,h)=0,$$
$$\mu\frac{d}{d\mu}\Phi=\beta^\Phi(G,B,\Phi,h)=0$$

at the level h^0 turn out to be Einstein Equations with 2-form field B and dilaton Φ:

$$R_{\mu\nu}=\frac{1}{4}H_\mu^{\lambda\rho}H_{\nu\lambda\rho}-2\nabla_\mu\nabla_\nu\Phi,$$
$$\nabla^\mu H_{\mu\nu\rho}-2(\nabla^\lambda\Phi)H_{\lambda\nu\rho}=0,$$
$$4(\nabla_\mu\Phi)^2-4\nabla_\mu\nabla^\mu\Phi+R+\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}=0,$$

where 3-form $H=dB$, and $R_{\mu\nu}, R$ are Ricci and scalar curvature correspondingly.

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In the early days of string theory:

Linearized Einstein Equations and their symmetries: $(G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu},\ B_{\mu\nu} = b_{\mu\nu},\ \Phi = \phi)$:

$$Q^{\eta}\Psi(s, b, \phi) = 0, \quad \Psi^{s}(s, b, \phi) \to \Psi(s, b, \phi) + Q^{\eta}\Lambda$$

in a semi-infinite complex associated to Virasoro module of Hilbert space of states for the "free" theory, asssociated to flat metric.

It was conjectured (A. Sen, B. Zwiebach,...) in the early 90s that Einstein equations with h-corrections are Generalized Maurer-Cartan (GMC) Equations:

$$Q^{\eta}\Psi + \frac{1}{2}[\Psi, \Psi]_h + \frac{1}{3!}[\Psi, \Psi, \Psi]_h + ... = 0$$

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In this talk:

i) Introducing complex structure:

Proper chiral "free action" $\rightarrow$ sheaves of vertex algebras/vertex algebroids.

Metric, B-field $\rightarrow$ Beltrami-Courant differential.

ii) Vertex algebroids $\rightarrow$ G_∞-algebras (G stands for Gerstenhaber).

Quasiclassical limit:
vertex algebroid $\rightarrow$ Courant algebroid, G_∞ algebra is truncated.

iii) Einstein equations and their h-corrections via Generalized Maurer-Cartan equation for L_∞-subalgebra of $G_\infty \otimes \bar{G}_\infty$.

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First order version of sigma-model action

We start from the action functional:

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma \mathcal{L}_0, \quad \mathcal{L}_0 = \langle p\wedge\bar\partial X\rangle - \langle\bar p\wedge\partial X\rangle,$$

where p, $\bar p$ are sections of $X^*(\Omega^{(1,0)}(M))\otimes\Omega^{(1,0)}(\Sigma)$, $X^*(\Omega^{(0,1)}(M))\otimes\Omega^{(0,1)}(\Sigma)$ correspondingly.

Infinitesimal local symmetries:

$$\mathcal{L}_0 \to \mathcal{L}_0 + d\xi$$

For holomorphic transformations we have:

$$X^i \to X^i - v^i(X), X^{\bar i} \to X^{\bar i} - v^{\bar i}(\bar X),$$
$$p_i \to p_i + \partial_i v^k p_k, \quad p_{\bar i} \to p_{\bar i} + \partial_{\bar i} v^{\bar k} p_{\bar k}$$
$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k), \quad p_{\bar i} \to p_{\bar i} - \bar\partial X^{\bar k}(\partial_{\bar k}\omega_{\bar i} - \partial_{\bar i}\omega_{\bar k}).$$

Not invariant under general diffeomorphisms, i.e.

$$\delta\mathcal{L}_0 = -\langle\bar\partial v, p\wedge\bar\partial X\rangle + \langle\partial\bar v, \bar p\wedge\partial X\rangle.$$

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$$\delta\mathcal{L}_\mu = -\langle \mu, p \wedge \bar{\partial} X\rangle - \langle \bar{\mu}, \partial X \wedge \bar{p}\rangle,$$

where $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}(M))$, $\bar{\mu} \in \Gamma(T^{(0,1)}M \otimes T^{*(1,0)}(M))$, so that: $\mu \to \mu - \bar{\partial} v + \dots$, $\bar{\mu} \to \bar{\mu} - \partial \bar{v} + \dots$.

Continuing the procedure:

$$\tilde{\mathcal{L}} = \langle p \wedge \bar{\partial} X\rangle - \langle \bar{p} \wedge \partial X\rangle - \\ \langle \mu, p \wedge \bar{\partial} X\rangle - \langle \bar{\mu}, \partial X \wedge \bar{p}\rangle - \langle b, \partial X \wedge \bar{\partial} X\rangle,$$

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so that the transformations of X- and p- fields are:

$$X^i \to X^i - v^i(X, \bar{X}), \quad p_i \to p_i + p_k \partial_i v^k - p_k \mu^k_{\bar{l}} \partial_i v^{\bar{l}} - b_{j\bar{k}} \partial_i v^{\bar{k}} \partial X^j, \\ X^{\bar{i}} \to X^{\bar{i}} - v^{\bar{i}}(X, \bar{X}), \quad \bar{p}_{\bar{i}} \to \bar{p}_{\bar{i}} + \bar{p}_{\bar{k}} \partial_{\bar{i}} v^{\bar{k}} - \bar{p}_{\bar{k}} \bar{\mu}^{\bar{k}}_l \partial_i v^l - b_{\bar{j}k} \partial_{\bar{i}} v^k \bar{\partial} X^{\bar{j}}.$$

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$$X^{\bar i} \to X^{\bar i} - v^{\bar i}(X,\bar X), \quad \bar p_{\bar i} \to \bar p_{\bar i} + \bar p_{\bar k}\partial_{\bar i} v^{\bar k} - \bar p_{\bar k}\bar\mu^{\bar k}_l\partial_i v^l - b_{\bar j k}\partial_{\bar i} v^k\bar\partial X^{\bar j}.$$

Similarly, for the 1-form transformation we obtain:

$$b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu^{i}_{\bar{j}}(\partial_i\omega_k - \partial_k\omega_i) +$$
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and

$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_{k}\partial_i\omega_{\bar{s}}\partial X^k,$$
$$p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial}X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial}X^r - \mu^{s}_{\bar{k}}\partial_i\omega_s\bar{\partial}X^{\bar{k}}.$$

For simplicity:

$$E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}},$$
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Let $\tilde{\mathbb{M}} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$\tilde{\mathbb{M}} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

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$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$
$$\tilde{\mathbb{M}} = \sum_I a^I \otimes \bar{a}^I,$$

where $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$, $f^I \in J^\infty(\mathcal{O}_M)$ and $\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$, $\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)$. Then

$$\phi_1(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where $[\cdot,\cdot]_D$ is a *Dorfman bracket*:

$$[v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v \omega,$$
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Courant bracket is the antisymmetrized version of $[\cdot,\cdot]_D$.
Similarly:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}}$$
$$\frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J (\bar{f}^I) \bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J (f^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

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V.N. Popov, M.G. Zeitlin, Phys.Lett. B 163 (1985) 185, A. Losev, A. Marshakov, A.Z., Phys. Lett. B 633 (2006) 375

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The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on M) via sheaves of VOA on M (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

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$$V=\mathrm{Span}\{p_{j_1,-s_1},\dots,p_{j_k,-s_k}X^{i_1}_{-r_1}\dots X^{i_l}_{-r_l}\}\otimes F(U)\otimes \mathbb{C}[h,h^{-1}],$$
$$r_m, s_n > 0,$$

$F(U)$ generated by X^i_0-modes.

Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on M) via sheaves of VOA on M (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set U of M we have VOA:

$$V=\sum_{n=0}^{\infty} V_n, \quad Y: V \to End(V)[[z,z^{-1}]],$$

generated by:

$$[X^i(z), p_j(w)] = h\delta^i_j\delta(z-w), \quad i,j=1,2,\dots,D/2$$
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The Virasoro element is:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{h} : \langle p(z)\partial X(z)\rangle : + \partial^2\phi'(X(z)).$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}(n^3 - n)\delta_{n,-m}$$

corresponding to correction:

$$\mathcal{L}_0 \to \mathcal{L}_{\phi'} = \langle p \wedge \bar{\partial} X\rangle - 2\pi i h R^{(2)}(\gamma)\phi'(X)$$

where $\phi' = \log\Omega$, where $\Omega(X)dX^1 \wedge \dots \wedge dX^n$ is a holomorphic volume form, i.e. for globally defined $T(z)$, M has to be Calabi-Yau.
The space V is a lowest weight module for the above Virasoro algebra.

V can be reproduced from V_0 and V_1 as a *vertex envelope*. The structure of vertex algebra imposes algebraic relations on $V_0 \oplus V_1$ giving it a structure of a *vertex algebroid*.

In our case: $V_0 \to \mathcal{O}^h_U = \mathcal{O}_U \otimes \mathbb{C}[h, h^{-1}]$,
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The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1} = c(z)T(z) + :c(z)\partial c(z) b(z):$$

is nilpotent when $D = 26$ (famous dimension 26!). However, we will consider subcomplex of light modes (i.e. $L_0 = 0$) denoted in the following as $(\mathcal{F}^{\cdot}_h, Q)$, where we can drop this condition:

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$$\Lambda \quad \text{generated by} \quad [b(z), c(w)]_+ = \delta(z-w).$$

The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1} = c(z)T(z) + :c(z)\partial c(z) b(z):$$

is nilpotent when $D = 26$ (famous dimension 26!). However, we will consider subcomplex of light modes (i.e. $L_0 = 0$) denoted in the following as $(\mathcal{F}^{\cdot}_h, Q)$, where we can drop this condition:

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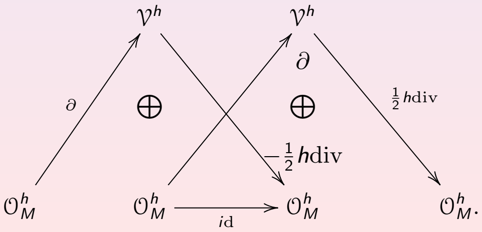

The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

$$(A, B)_h = Res_z \frac{A(z)B}{z}$$

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\begin{aligned}
&Q(a_1, a_2)_h = (Qa_1, a_2)_h + (-1)^{|a_1|}(a_1, Qa_2)_h,\\
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Operator $\mathbf{b}$ of degree -1 (0-mode of $b(z)$) on $(\mathcal{F}_h^{\cdot}, Q)$ which anticommutes with Q:

$$
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 & \mathcal{V}^h & \xleftarrow{-id} & \mathcal{V}^h & \\
 & \bigoplus & & \bigoplus & \\
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$$(-1)^{|a_1|}\{a_1, a_2\}_h = \mathbf{b}(a_1, a_2)_h - (\mathbf{b}a_1, a_2)_h - (-1)^{|a_1|}(a_1\mathbf{b}a_2)_h,$$

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$$\{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\}_h =$$
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The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to G_∞-algebra.

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Homotopy algebras: G_∞, L_∞, C_∞

Let A be a graded vector space, consider free graded Lie algebra $Lie(A)$.

$$Lie^{k+1}(A) = [A, Lie^k A], \quad Lie^1(A) = A.$$

Consider free graded commutative algebra GA on the suspension $(Lie(A))[-1]$, i.e.

$$GA = \oplus_n \bigwedge^n Lie(A)[-n]$$

There are natural $[\cdot,\cdot]$, $\wedge$ operations on GA of degree -1, 0 correpondingly, generating a Gerstenhaber algebra.

A G_∞-*algebra* (Tamarkin, Tsygan, 2000) is a graded space V with a differential ∂ of degree 1 of $G(V[1]^*)$, such that ∂ is a derivation w.r.t bracket and the product.

Multiplicative Ideals, preserved by ∂: I_1-generated by the commutant of $Lie(V[1]^*)$, $I_2 = \bigwedge_{n\geq 2}(Lie(V[1]^*)[-n]$. That induces differentials on corresponding factors: $\bigwedge_{n\geq 1}(V[1]^*)[-n]$ and $Lie(V[1]^*)[-1]$. The resulting structures on V are called L_∞-*algebra* and C_∞-*algebra* correspondingly.

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Restriction of ∂ on $V[1]^*$:

$$V[1]^* \to Lie^{k_1}(V[1]^*) \wedge \cdots \wedge Lie^{k_n}(V[1]^*).$$

Conjugated map:

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of degree $3 - n - k_1 - \ldots - k_n$, satisfying bilinear relations.

In our previous notation $m_1 = Q$, m_2-symmetrized LZ product, $m_{1,1}$–antisymmetrized LZ bracket.

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An important feature of L_∞ algebra is a Maurer-Cartan equation (Φ is of degree 2) :

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of degree $3 - n - k_1 - \ldots - k_n$, satisfying bilinear relations.

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Quasiclassical limit of LZ G_∞ algebra

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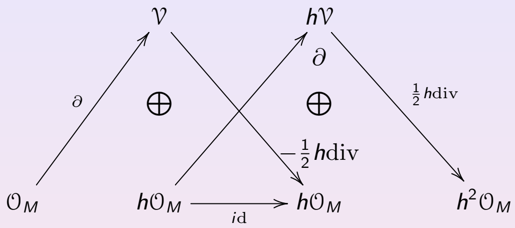

is a subcomplex of $(\mathcal{F}^{\cdot}_h, Q)$. Then

$$(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\}_h : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h],$$
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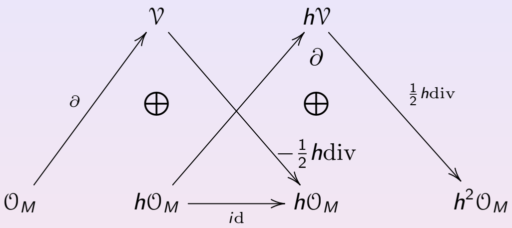

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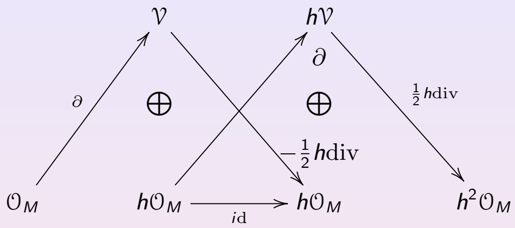

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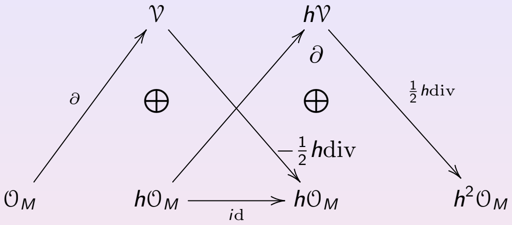

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The symmetrized operations $(\cdot,\cdot)_0$, $\{\cdot,\cdot\}_0, \ldots$ satisfy the relations of the homotopy Gerstenhaber algebra, so that all non-covariant higher-order terms disappear from the multilinear operations.

The resulting C_∞ and L_∞ algebras are reduced to C_3 and L_3 algebras.

A.Z., Comm. Math. Phys. 303 (2011) 331-359.

Conjecture: This G_∞-algebra is the G_3-algebra (no homotopies beyond trilinear operations).

Classical limit procedure for vertex algebroid (due to P. Bressler): $[\cdot,\cdot]_0 = \lim_{h\to 0}\frac{1}{h}[\cdot,\cdot]$, $\pi_0 = \lim_{h\to 0}\frac{1}{h}\pi$, $\langle\cdot,\cdot\rangle_0 = \lim_{h\to 0}\frac{1}{h}\langle\cdot,\cdot\rangle$.

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A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with a structure of a Leibniz $\mathbb{C}$-algebra $[\cdot,\cdot]_0 : \mathcal{Q}\otimes_{\mathbb{C}}\mathcal{Q}\to\mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0 : \mathcal{Q}\to\Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{Q}\otimes_{\mathcal{O}_M}\mathcal{Q}\to\mathcal{O}_M$, a derivation $\partial : \mathcal{O}_M\to\mathcal{Q}$ which satisfy

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In our case $\mathcal{Q}\cong\mathcal{O}(\mathcal{E})$, π_0 is just a projection on $\mathcal{O}(TM)$

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We show that it is a part of a more general structure, homotopy Gerstenhaber algebra.

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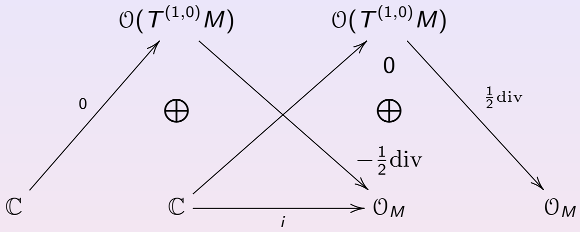
Simplest version: $G_\infty \to$ Gerstenhaber algebra

Subcomplex $(\mathcal{F}^{\cdot}_{sm}, Q)$:



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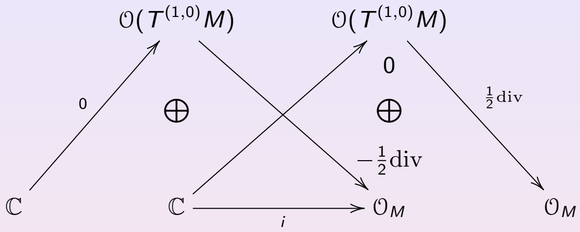

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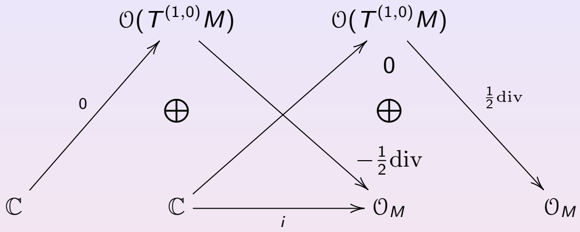

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Maurer-Cartan elements, closed under $\mathbf{b}^-$:

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Components:$(g, \bar{v}, v, \phi, \bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

1). Vector field $div_\Omega g$, where $\log \Omega = -2\Phi_0 = -2(\phi' + \bar{\phi}' + \phi + \bar{\phi})$ and $\partial_i \partial_{\bar{j}} \Phi_0 = 0$, is such that its $\Gamma(T^{(1,0)}M)$, $\Gamma(T^{(0,1)}M)$ components are correspondingly holomorphic and antiholomorphic.
2). Bivector field $g \in \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$ obeys the following equation:

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Maurer-Cartan elements, closed under $\mathbf{b}^-$:

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Components:$(g, \bar{v}, v, \phi, \bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

1). Vector field $div_\Omega g$, where $\log \Omega = -2\Phi_0 = -2(\phi' + \bar{\phi}' + \phi + \bar{\phi})$ and $\partial_i \partial_{\bar{j}} \Phi_0 = 0$, is such that its $\Gamma(T^{(1,0)}M)$, $\Gamma(T^{(0,1)}M)$ components are correspondingly holomorphic and antiholomorphic.

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These are Einstein equations with the following constraints:

$$G_{i\bar{k}} = g_{i\bar{k}}, \quad B_{i\bar{k}} = -g_{i\bar{k}}, \quad \Phi = \log\sqrt{g} + \Phi_0,$$
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Physically:

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Consider

$$\mathbf{F}^{\cdot}_{b^-} = \mathcal{F}^{\cdot} \otimes \bar{\mathcal{F}}^{\cdot}|_{b^-=0}$$

with the L_∞-algebra structure given by Lian-Zuckerman construction.

One can explicitly check that GMC symmetry
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$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \ldots,$$

reproduces

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}).$$

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