$\hbar$-opers and the geometric approach to the Bethe ansatz

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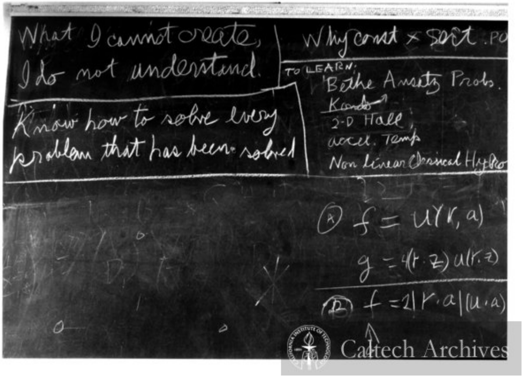
Stony Brook

May 31, 2022



R.P. Feynman: “I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you dont know why. I am trying to understand all this better.”

Various points of view on Bethe ansatz

- via Algebraic Bethe ansatz:
  Central for the QISM.
  Developed in Leningrad: late 70s-80s
- via Frenkel-Reshetikhin (qKZ) equation:
  I. Frenkel, N. Reshetikhin '92
  Recently: geometrization through enumerative geometry of quiver varieties.
  A. Okounkov '15; A. Okounkov, A. Smirnov '16; M. Aganagic, A. Okounkov '17;
  P. Pushkar, A. Smirnov, A.Z. '16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z .'17
- via QQ-systems:
  appeared first in the context of qKdV equation and ODE/IM correspondence
  V. Bazhanov, S. Lukyanov, A. Zamolodchikov'98; D. Masoero, A. Raimondo, D. Valeri'16; Frenkel, Hernandez '13,'19
  In this talk: geometric interpretation of QQ-systems through the difference analogue of connections on the projective line, the so-called $(G, \hbar)$-opers.

Based on joint work with E. Frenkel, P. Koroteev, D. Sage '18 – '22

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QQ-systems

Consider Lie algebra $\mathfrak{g}$ of rank r.

Cartan matrix: $\{a_{ij}\}_{i,j=1,\dots,r}$, $a_{ij}=\langle\check{\alpha}_i,\alpha_j\rangle$.

QQ-system:

$$\begin{aligned}\widetilde{\xi}_i Q^i_-(u)Q^i_+(\hbar u)-\xi_i Q^i_-(\hbar u)Q^i_+(u) &= \Lambda_i(u)\prod_{j\neq i}\left[\prod_{k=1}^{-a_{ij}} Q^j_+(\hbar^{b^k_{ij}}u)\right]\\ i &= 1,\dots,r, \quad b^k_{ij}\in\mathbb{Z}\end{aligned}$$

$\{\Lambda_i(u), Q^i_\pm(u)\}_{i=1,\dots,r}$– polynomials, ξ_i, $\widetilde{\xi}_i$, $\hbar\in\mathbb{C}^\times$;
$\{\Lambda_i(z)\}_{i=1,\dots,r}$-fixed.

Solving for $\{Q^i_+(z)\}_{i=1,\dots,r}$; $\{Q^i_-(z)\}_{i=1,\dots,r}$-auxiliary.

$$\text{If } \mathfrak{g} \text{ is of ADE type}: \begin{cases} b_{ij}=1, & i>j\\ b_{ij}=0, & i<j\end{cases}$$

Example: $\mathfrak{g}=\mathfrak{sl}(2)$:

$$\widetilde{\xi}Q_-(u)Q_+(\hbar u)-\xi Q_-(\hbar u)Q_+(u)=\Lambda(u).$$

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- Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\widehat{\mathfrak{g}})$.

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; E. Frenkel, D. Hernandez '13,'19

- Bethe ansatz equations for XXX, XXZ models: $Q^i_\pm$ are eigenvalues of Baxter operators.

  in case $\xi_i, \widetilde{\xi}_i = 1$: E. Mukhin, A. Varchenko, ...

- Relations in quantum equivariant K-theory, quantum cohomology of quiver varieties Baxter operators are generating functions of tautological bundles $\widehat{Q}^i_+(u) = \sum_{m=0}^{n} u^m \Lambda^m \mathcal{V}_i$.

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- Spectral determinant relations in ODE/IM correspondence

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Simple patterns in representation theory

- $\{V_{\omega_i}\}_{i=1,\dots,r}$ – fundamental representations of $\mathfrak{g}$. Homomorphisms m_i:

$$m_i : \quad \Lambda^2 V_{\omega_i} \quad \to \quad \otimes_{j\neq i} V_{\omega_j}^{\otimes^{-a_{ji}}}$$

This is how QQ-system appears in ODE/IM correspondence (D. Masoero, A. Raimondo, D. Valeri '16)

- Relations between generalized minors:

Lewis Carroll identity:

$$det(M_1^1)det(M_k^k) - det(M_1^k)det(M_k^1) = det(M)\ det(M_{1,k}^{1,k})$$

More generally (S. Fomin, A. Zelevinsky '98):

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}(g)\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i}(g) \quad - \quad \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}(g)\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i}(g) = \prod_{j\neq i}\Big[\Delta_{u\cdot\omega_j, v\cdot\omega_j}(g)\Big]^{-a_{ji}}.$$

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$$\left[q^i_+(v)\partial_v q^i_-(v) - q^i_-(v)\partial_v q^i_+(v)\right] + \zeta_i q_i^+(v) q_i^-(v) = \Lambda_i(v)\prod_{j\neq i}\left[q^j_+(v)\right]^{-a_{ji}}$$

$$i = 1, \dots, r$$

for $\mathfrak{g}$ with Cartan matrix $\{a_{ji}\}_{i,j=1,\dots,r}$.

We will retell a version of a classic story between oper connections on the projective line and Gaudin models:

E. Frenkel'03; B. Feigin, E. Frenkel, V. Toledano-Laredo '06,

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One-to-one correspondence (with some nondegeneracy conditions):

Polynomial solutions to the ***qq*****-system**

$\updownarrow$

Miura ***G*****-oper connections on $\mathbb{P}^1$ with regular singularities, trivial monodromy and the double pole at infinity**

T. Brinson, D. Sage, A.Z. '21

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Differential limit: qq-system

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Miura oper connections

Miura oper connections on $\mathbb{P}^1$ as a differential operator:

$$\nabla_v = \partial_v + \sum_{i=1}^{r} \zeta_i \check{\omega}_i - \sum_{i=1}^{r} \partial_v \log[q_i^+(v)]\check{\alpha}_i + \sum_{i=1}^{r} \Lambda_i(v) e_i .$$

Here

$$\Lambda_i(v) = \prod_{k=1}^{N} (v - v_k)^{\langle \alpha_i, \check{\lambda}_k \rangle},$$

v_k–are known as regular singularities;

$$q_+^i(v) = \prod_k (v - w_k^i).$$

$\mathcal{Z}$-twisted condition:

$$\nabla_v = U(v)(\partial_v + \mathcal{Z})U(v)^{-1}, \quad \mathcal{Z} = \sum_{i=1}^{r} \zeta_i \check{\omega}_i$$

$$U(v) = \prod_{i=1}^{r} [q_+^i(v)]^{\check{\alpha}_i} \prod_{j=1}^{r} \exp\left[- \frac{q_-^i(v)}{q_+^i(v)} e_i \right] \dots$$

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qq-system for $\mathfrak{g}$ $\leftrightarrow$ ${}^L\mathfrak{g}$ – Gaudin model Bethe equations

Bethe equations for the Gaudin model:

$$\sum_{i=1}^{N}\frac{\langle\check{\lambda}_i,\alpha_{k_j}\rangle}{w_j-v_i}-\sum_{s\neq j}\frac{\langle\check{\alpha}_{i_s},\alpha_{k_j}\rangle}{w_j-w_s}=\zeta_{k_j},\quad j=1,\dots,m.$$

Commuting Gaudin Hamiltonians:

B. Feigin, E. Frenkel, V. Toledano-Laredo '06, E. Frenkel, L. Rybnikov '07

$$H_i=\sum_{k\neq i}\sum_{a=1}^{\dim {}^L\mathfrak{g}}\frac{x_a^{(i)}x_a^{(k)}}{v_i-v_k}+\sum_{a=1}^{\dim {}^L\mathfrak{g}}\mu(x_a)x_a^{(i)}$$

acting on

$$V_{\check{\lambda}_1}\otimes V_{\check{\lambda}_2}\otimes\cdots\otimes V_{\check{\lambda}_N}.$$

Here $\mu\in({}^L\mathfrak{g})^*$ is regular semisimple.

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Bethe equations for the Gaudin model:

$$\sum_{i=1}^{N} \frac{\langle \check{\lambda}_i, \alpha_{k_j} \rangle}{w_j - v_i} - \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_s}, \alpha_{k_j} \rangle}{w_j - w_s} = \zeta_{k_j}, \quad j = 1, \dots, m.$$

Commuting Gaudin Hamiltonians:

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$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim {}^L\mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{v_i - v_k} + \sum_{a=1}^{\dim {}^L\mathfrak{g}} \mu(x_a) x_a^{(i)}$$

acting on

$$V_{\check{\lambda}_1} \otimes V_{\check{\lambda}_2} \otimes \cdots \otimes V_{\check{\lambda}_N}.$$

Here $\mu \in ({}^L\mathfrak{g})^*$ is regular semisimple.

Elementary example: SL(2)-oper

GL(2)-oper:

- Triple: $(E, \nabla, \mathcal{L})$ on $\mathbb{P}^1$:
  E-vector bundle, rank(E)=2, $\mathcal{L}$-line subbundle, ∇–connection.
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It is an $SL(2)$-oper if $GL(2)$ can be reduced to $SL(2)$.

Locally, second condition: $s(v) \wedge \nabla_v s(v) \neq 0$,
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$SL(2)$-oper with regular singularities: $s(v) \wedge \nabla_v s(v) \sim (v - v_i)^{k_i}$ near v_i.

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Thus the oper condition is:

$$s(v) \wedge (\partial_v + \mathcal{Z}) s(v) = \Lambda(v),$$

where $\Lambda(v) \sim \prod_i (v - v_i)^{k_i}$, $\quad \mathcal{Z} = \begin{pmatrix} \zeta/2 & 0 \\ 0 & -\zeta/2 \end{pmatrix}$.

Explicitly: $s(v) = \begin{pmatrix} q_-(v) \\ q_+(v) \end{pmatrix}$, we have:

$$q_+(v)\partial_v q_-(v) - q_-(v)\partial_v q_+(v) + \zeta q_+(v) q_-(v) = \Lambda(v).$$

Rewriting:

$$\partial_v \left[- e^{-\zeta v} \frac{q_-(v)}{q_+(v)} \right] = \frac{e^{-\zeta v} \Lambda(v)}{q_+(v)^2}$$

and computing residues, obtain $\mathfrak{sl}(2)$ Gaudin Bethe ansatz equations:

$$-\zeta + \sum_{n=1}^{N} \frac{k_n}{v_n - w_i} = \sum_{j \neq i} \frac{2}{w_j - w_i}.$$

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SL(2)-oper and Bethe equations

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SL(2) Miura oper

Introduce line bundle $\hat{\mathcal{L}}$ preserved by ∇.

Miura oper is a quadruple:

$$(E, \nabla, \mathcal{L}, \hat{\mathcal{L}}).$$

Choose trivialization of E so that:

$$\hat{s}(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad s(v) = \begin{pmatrix} q_-(v) \\ q_+(v) \end{pmatrix}$$

These are sections, generating $\hat{\mathcal{L}}$ and $\mathcal{L}$ correspondingly.

Notice that $\mathcal{L}, \hat{\mathcal{L}}$ span E except for points corresponding to Bethe roots.

Standard form of Miura oper

Choosing upper-triangular $g(v)$, such that $g(v)s(v) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$g(v) = \begin{pmatrix} q_+(v) & -q_-(v) \\ 0 & q_+(v)^{-1} \end{pmatrix}$$

we obtain Miura oper connection in the standard form:

$$\nabla_v = \partial_v + g(v)\partial_v g(v)^{-1} + g(v)\mathcal{Z}g(v)^{-1} =$$
$$\partial_v + \begin{pmatrix} \zeta/2 - \partial_v \log[q_+(v)] & \Lambda(v) \\ 0 & -\zeta/2 + \partial_v \log[q_+(v)] \end{pmatrix}$$

Or, in other words, we obtained the standard for of Miura oper connection, we have seen before:

$$\partial_v + \mathcal{Z} - \partial_v \log[q_+(v)]\check{\alpha} + \Lambda(v)e$$

$SL(r+1)$-opers

GL(r+1)-opers:

Triple: $(E, \nabla, \mathcal{L}_\bullet)$, rank$(E)$=r+1, ∇–connection,

$\mathcal{L}_\bullet$– flag of subbundles:

- $\nabla : \mathcal{L}_i \to \mathcal{L}_{i+1} \otimes K$
- induced map $\overline{\nabla}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \to \mathcal{L}_{i+1}/\mathcal{L}_i \otimes K$ is an isomorphism.

If structure group reduces to $SL(r+1)$, the above triple gives $SL(r+1)$-opers.

Locally, oper condition can be reformulated as:

$$0 \neq W_i(s)(v) = (s(v) \wedge \nabla_v s(v) \wedge \cdots \wedge \nabla_v^{i-1} s(v))|_{\Lambda^i \mathcal{L}},$$

where $s(v)$ is a section of $\mathcal{L}_1$.

Regular singularities: relaxing these conditions, by adding zeroes for $W_i(s)$.

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$SL(r+1)$ Miura opers and qq-system

Oper connection with regular singularities as a matrix:

$$\nabla_v=\partial_v+\begin{pmatrix} * & \Lambda_1(v) & 0 & \dots & 0\\ * & * & \Lambda_2(v) & 0\dots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ * & * & \dots & * & \Lambda_r(v)\\ * & * & * & * & * \end{pmatrix}$$

Miura oper: quadrupe $(E,\nabla,\mathcal{L}_\bullet,\hat{\mathcal{L}}_\bullet)$.

Here ∇ preserves another flag of subbundles: $\hat{\mathcal{L}}_\bullet$:

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$(SL(2),\hbar)$-connection

$$M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1, \text{ such that } u \to \hbar u.$$

Bundle $E \to \mathbb{P}^1$, rank(E)=2, $\quad E^\hbar \to \mathbb{P}^1$ is a pull-back bundle.

$(SL(2),\hbar)$-connection: A is a meromorphic section of

$$Hom_{\mathcal{O}_{\mathbb{P}^1}}(E, E^\hbar),$$

so that $A(u) \in SL(2, \mathbb{C}(u))$.

$\hbar$-gauge transformations:

$$A(u) \to g(\hbar u) A(u) g^{-1}(u)$$

$(SL(2),\hbar)$-oper

$(SL(2),\hbar)$-oper on $\mathbb{P}^1$ with regular singularities is a triple $(E,A,\mathcal{L})$:

- (E,A) is a $(SL(2),\hbar)$-connection
- $\mathcal{L}$ is a line subbundle so that $\bar{A}:\mathcal{L}\to(E/\mathcal{L})^{\hbar}$ is an isomorphism

Locally:

$$s(\hbar u)\wedge A(u)s(u)\neq 0,$$

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Regular singularities: $\Lambda(u)=\prod_{m=1}^{N}\prod_{j=0}^{k_m-1}(u-\hbar^{-j}u_m)$, so that:

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Miura $(SL(2), \hbar)$-oper and the QQ-system

Given that $s(u) = \begin{pmatrix} Q_-(u) \\ Q_+(u) \end{pmatrix}$, the condition $s(\hbar u) \wedge Z s(u) = \Lambda(u)$ is equivalent to:

$$z Q_+(\hbar u) Q_-(u) - z^{-1} Q_-(\hbar u) Q_+(u) = \Lambda(u)$$

Bethe equations for XXZ model:

$$\frac{\Lambda(w_i)}{\Lambda(\hbar^{-1} w_i)} = -z^2 \frac{Q_+(\hbar w_i)}{Q_+(\hbar^{-1} w_i)}$$

$$Q_+(u) = \prod_j (u - w_j)$$

Canonical form of Miura $(SL(2), \hbar)$-oper

Considering $U(u)s(u) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so that $\hat{\mathcal{L}}$ is preserved, gives:

$$U(u) = \begin{pmatrix} Q_+(u) & -Q_-(u) \\ 0 & Q_+(u)^{-1} \end{pmatrix}$$

which leads to:

$$A(u) = U(\hbar u) Z U(u)^{-1} = \begin{pmatrix} z \frac{Q_+(\hbar u)}{Q_+(u)} & \Lambda(u) \\ 0 & z^{-1} \frac{Q_+(u)}{Q_+(\hbar u)} \end{pmatrix}.$$

In universal terms:

$$A(u) = g^{\check{\alpha}}(u) e^{\frac{\Lambda(u)}{g(u)} e}, \quad g(u) = z \frac{Q_+(\hbar u)}{Q_+(u)}.$$

Compare to the Miura SL(2)-oper connection:

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Quantum group

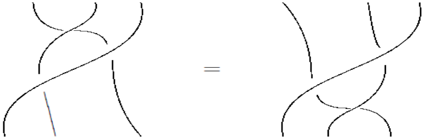
$$U_\hbar(\hat{\mathfrak{g}})$$

is a deformation of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1) \otimes V_2(a_2)$$

$$V_2(a_2) \otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:



The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices (the so-called FRT construction).

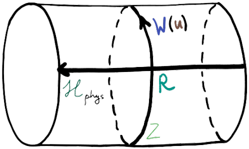
Transfer matrices

Source of integrability: commuting *transfer matrices*, generating *Baxter algebra* which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}}: W(u)\otimes\mathcal{H}_{\rm phys}\to W(u)\otimes\mathcal{H}_{\rm phys}$$

over auxiliary $W(u)$ space:

$$T_{W(u)}=\mathrm{Tr}_{W(u)}\Big(M(u)\Big)=\mathrm{Tr}_{W(u)}\Big((Z\otimes 1)\ \tilde{R}_{W(u),\mathcal{H}_{phys}}\Big)$$



Here $Z\in e^{\mathfrak{h}}$, where $\mathfrak{h}\subset\mathfrak{g}$ is a Cartan subalgebra.

Integrability and Baxter algebra

Integrability condition:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

There are special transfer matrices is called *Baxter Q-operators*. Such operators generate all *Bethe algebra*.

Primary goal for physicists is to diagonalize $\{T_{W(u)}\}$ simultaneously.

$(G,\hbar)$-connections on $\mathbb{P}^1$

- Principal G-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$
- $M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1$, such that $u \mapsto \hbar u$.

  $\mathcal{F}_G^\hbar$ stands for the pullback under the map $M_\hbar$.

A meromorphic $(G,\hbar)$-connection on a principal G-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ is a section A of $Hom_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^\hbar)$, where U is a Zariski open dense subset of $\mathbb{P}^1$.

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to the trivial G-bundle.

The restriction of A to the Zariski open dense subset $U \cap M_\hbar{}^{-1}(U)$ is an element $A(u)$ of $G(u) \equiv G(\mathbb{C}(u))$.

Changing the trivialization is given by $\hbar$-gauge transformation:

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  $\mathcal{F}_G^\hbar$ stands for the pullback under the map $M_\hbar$.

A meromorphic $(G, \hbar)$-connection on a principal G-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ is a section A of $Hom_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^\hbar)$, where U is a Zariski open dense subset of $\mathbb{P}^1$.

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to the trivial G-bundle.

The restriction of A to the Zariski open dense subset $U \cap {M_\hbar}^{-1}(U)$ is an element $A(u)$ of $G(u) \equiv G(\mathbb{C}(u))$.

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$(G,\hbar)$-oper connections for simple simply connected Lie groups G

A $(G,\hbar)$-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$:

- $\mathcal{F}_G$ is a G-bundle
- A is a meromorphic $(G,\hbar)$-connection on $\mathcal{F}_G$ over $\mathbb{P}^1$
- $\mathcal{F}_{B_-}$ is the reduction of $\mathcal{F}_G$ to B_-

$(G,\hbar)$-oper condition: restriction of the connection $A: \mathcal{F}_G \to \mathcal{F}_G^{\hbar}$ to $U \cap {M_\hbar}^{-1}(U)$ takes values in the Bruhat cell

$$B_-(\mathbb{C}[U \cap {M_\hbar}^{-1}(U)])\, c\, B_-(\mathbb{C}[U \cap {M_\hbar}^{-1}(U)]),$$

where c is Coxeter element: $c = \prod_i s_i$.

Locally:

$$A(u) = n'(u)\prod_i \left[\phi_i(u)^{\check{\alpha}_i} s_i\right] n(u),\ \ \phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N(u)$$

Here $N = B/H$, $H = B/[B,B]$.

$\hbar$-Drinfeld-Sokolov reduction: Semenov-Tian-Shansky, Sevostyanov '98

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$\hbar$-Drinfeld-Sokolov reduction: Semenov-Tian-Shansky, Sevostyanov '98

A Miura $(G, \hbar)$-oper on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$:

- $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic $(G, \hbar)$-oper on $\mathbb{P}^1$.
- $\mathcal{F}_{B_+}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_+ that is preserved by the $(G, \hbar)$-connection A.

The fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at x is a G-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+, respectively. Choose any trivialization of $\mathcal{F}_{G,x}$, i.e. an isomorphism of G-torsors $\mathcal{F}_{G,x} \simeq G$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with bB_+.

Then $a^{-1}b$ is a well-defined element of the double quotient $B_- \backslash G / B_+$, which is in bijection with W_G.

We will say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have a generic relative position at $x \in X$ if the element of W_G assigned to them at x is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_- \cdot B_+ \subset G$).

Theorem.
i) For any Miura $(G, \hbar)$-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper $\hbar$-connection has the form:

$$A(u) \in N_-(u) \prod_i (\phi_i(u)^{\check{\alpha}_i} s_i) N_-(u) \, \cap \, B_+(u).$$

ii) Any element from $N_-(u) \prod_i (\phi_i(u)^{\check{\alpha}_i} s_i) N_-(u) \, \cap \, B_+(z)$ can be written as:

$$\prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\phi_i(u) t_i(u)}{g_i(u)} e_i}$$

where each $t_i \in \mathbb{C}(u)$ is determined by the lifting of s_i.

In the following we set $t_i \equiv 1$.

E. Frenkel, P. Koroteev, D. Sage, A.Z. '20

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E. Frenkel, P. Koroteev, D. Sage, A.Z. '20

$(G,\hbar)$-opers with regular singularities and Z-twisted opers

- $(G,\hbar)$-oper with regular singularities at finitely many points on $\mathbb{P}^1$:

$$A(u) = n'(u)\prod_i \left[\Lambda_i^{\check{\alpha}_i}(u)s_i\right]n(u),\ \ \Lambda_i(u)\in\mathbb{C}[u].$$

For any Miura $(G,\hbar)$-oper with regular singularities:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i}.$$

- $(G,\hbar)$-oper is Z-twisted if it is gauge equivalent to $Z\in H$, namely

$$A(u) = v(\hbar u)Zv^{-1}(u),\ \ \text{where } Z=\prod_i z_i^{\check{\alpha}_i},\ \ v(u)\in G(u).$$

We assume Z is regular semisimple. In that case there are W_G Miura opers for a given oper.

In the extreme case $Z=1$ we have G/B Miura opers for a given oper.

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Nondegeneracy conditions (see detailed discussion in our paper):

$$A(u)=\prod_i g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i},\quad g_i(u)=z_i\frac{y_i(\hbar u)}{y_i(u)}$$

Each $y_i(u)$ is a polynomial, and for all i,j,k with $i\neq j$ and $a_{ik}\neq 0$, $a_{jk}\neq 0$, the zeros of $y_i(u)$ and $y_j(u)$ are $\hbar$-distinct from each other and from the zeros of $\Lambda_k(u)$.

Explicit formula for $v(u)$, such that

$$A(u)=v(u\hbar)Zv(u)^{-1}$$

is:

$$v(u)=\prod_{i=1}^r y_i(u)^{\check{\alpha}_i}\prod_{i=1}^r e^{-\frac{Q^i_-(u)}{Q^i_+(u)}e_i}\dots,$$

where the dots stand for the exponentials of higher commutator terms.

Nondegenerate Z-twisted Miura $(G,\hbar)$-opers and QQ-systems

Nondegeneracy conditions (see detailed discussion in our paper):

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Main theorem

That leads to the expression of Miura $(G,\hbar)$-oper connection:

$$A(u)=\prod_i g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i}, \quad g_i(u)=z_i\frac{Q_+^i(\hbar u)}{Q_+^i(u)}.$$

Theorem. There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura $(G,\hbar)$-opers and the set of nondegenerate polynomial solutions of the QQ-system:

$$\widetilde{\xi}_i Q_-^i(u)Q_+^i(\hbar u)-\xi_i Q_-^i(\hbar u)Q_+^i(u)=$$
$$\Lambda_i(u)\prod_{j>i}\left[Q_+^j(\hbar u)\right]^{-a_{ji}}\prod_{j<i}\left[Q_+^j(u)\right]^{-a_{ji}}, \qquad i=1,\dots,r,$$

where $\widetilde{\xi}_i=z_i\prod_{j>i}z_j^{a_{ji}}$, $\xi_i=z_i^{-1}\prod_{j<i}z_j^{-a_{ji}}$.

E. Frenkel, P. Koroteev, D. Sage, A.Z. '20

In ADE case this QQ-system correspond to the Bethe ansatz equations. Beyond simply-laced case: "folded integrable models".

E. Frenkel, D. Hernandez, N. Reshetikhin '21

Originally operators

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q_+^i(\hbar u)}{Q_+^i(u)},$$

where $Q_\pm(u)$ are the solution of QQ-systems, were introduced by Mukhin, Varchenko'05 in the additive case with $Z = 1$.

They also introduced the following $\hbar$-gauge transformation of the $\hbar$-connection A:

$$A \mapsto A^{(i)} = e^{\mu_i(\hbar u) f_i} A(u) e^{-\mu_i(u) f_i}, \quad \text{where} \quad \mu_i(u) = \frac{\prod_{j \neq i} \left[Q_+^j(u)\right]^{-a_{ji}}}{Q_+^i(u) Q_-^i(u)}.$$

Then $A^{(i)}(u)$ can be obtained from $A(u)$ by substituting in formula for $A(u)$:

$$Q_+^j(u) \mapsto Q_+^j(u), \qquad j \neq i,$$
$$Q_+^i(u) \mapsto Q_-^i(u), \qquad Z \mapsto s_i(Z).$$

Altogether these transformation generate the "full" QQ-system.

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$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q_+^i(\hbar u)}{Q_+^i(u)},$$

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$SL(r+1)$ opers: explicit formula

QQ-system:

$$\xi_{i+1}Q_i^+(\hbar u)Q_i^-(u)-\xi_i Q_i^+(u)Q_i^-(\hbar u)=\Lambda_i(u)Q_{i-1}^+(u)Q_{i+1}^+(\hbar u)\,,\; i=1,\dots,r$$

$$\xi_1=\frac{1}{z_1},\quad \xi_2=\frac{z_1}{z_2},\quad \dots\quad \xi_r=\frac{z_{r-1}}{z_r},\quad \xi_{r+1}=\frac{1}{z_r}\,,$$

For Z-twisted oper:

$$A(u)=V^{-1}(\hbar u)ZV(u)$$

$$V(u)=\begin{pmatrix} \frac{1}{Q_1^+(u)} & \frac{Q_1^-(u)}{Q_2^+(u)} & \frac{Q_{12}^-(u)}{Q_3^+(u)} & \dots & \frac{Q_{1,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{1,\dots,r}^-(u) \\ 0 & \frac{Q_1^+(u)}{Q_2^+(u)} & \frac{Q_2^-(u)}{Q_3^+(u)} & \dots & \frac{Q_{2,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{2,\dots,r}^-(u) \\ 0 & 0 & \frac{Q_2^+(u)}{Q_3^+(u)} & \dots & \frac{Q_{3,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{3,\dots,r}^-(u) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \frac{Q_{r-1}^+(u)}{Q_r^+(u)} & Q_r^-(u) \\ 0 & \dots & \dots & \dots & 0 & Q_r^+(u) \end{pmatrix}.$$

$(SL(r+1),\hbar)$-opers: alternative definition

A $(GL(r+1),\hbar)$-*oper* on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

$$\mathcal{L}_{r+1} \subset ... \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset ... \subset E = \mathcal{L}_1,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that $A \in Hom_{\mathcal{O}_{\mathbb{P}^1}}(E, E^{\hbar})$ satisfies the following conditions:

- $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$,
- $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \to \mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism.

An $(SL(r+1),\hbar)$-*oper* is a $(GL(r+1),\hbar)$-oper with the condition that $det(A) = 1$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

$(SL(r+1),\hbar)$-Wronskians and QQ-systems

Minors in $\hbar$-Wronskian matrix:

$$\mathcal{D}_k(s) =$$
$$e_1 \wedge \cdots \wedge e_{r+1-k} \wedge s(u) \wedge Z^{-1}s(\hbar u) \wedge \cdots \wedge Z^{1-k}s(\hbar^{k-1}u) =$$
$$\alpha_k W_k(u)\mathcal{V}_k(u)\,,$$

where

$$\mathcal{V}_k(u) = \prod_{a=1}^{r_k}(u - w_{k,a})\,,$$

and

$$W_k(s) = P_1 \cdot P_2^{(1)} \cdot P_3^{(2)} \cdots P_{k-1}^{(k-2)}, \quad P_i = \Lambda_r\Lambda_{r-1}\cdots\Lambda_{r-i+1}$$

We used the notation $f^{(j)}(u) = f(\hbar^j u)$ above.

One can identify: $\mathcal{V}_k(u) \equiv Q_k^+(u)$ and $Q^-_{i,\ldots,j}(u)$ with other minors.

The bilinear relations for the extended QQ-system are nothing but Plücker relations for minors in the $\hbar$-Wronskian matrix.

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What about the analogue of $\hbar$-Wronskian for Miura $(G, \hbar)$-oper?

One can construct an analogue of the $\hbar$-Wronskian matrix as a solution of a difference equation, so that the full QQ-system emerge as relations for generalized minors.

P. Koroteev, A.Z.'21

Take section of the line bundle $\mathcal{L}_{r+1}$ in complete flag $\mathcal{L}_\bullet$:

$$s(u) = \begin{pmatrix} s_1(u) \\ s_2(u) \\ s_3(u) \\ \vdots \\ s_r(u) \\ s_{r+1}(u) \end{pmatrix} = \begin{pmatrix} Q^-_{1,\dots,r}(u) \\ Q^-_{2,\dots,r}(u) \\ Q^-_{3,\dots,r}(u) \\ \vdots \\ Q^-_r(u) \\ Q^+_r(u) \end{pmatrix}.$$

Interesting case (XXZ chain corresponding to defining representations):

- Polynomials are of degree 1
- Only $\Lambda_1(u) = \prod_i (u - a_i)$ is nontrival

Identification:

- roots of $s_i(u)$ with momenta p_i
- $\xi_i = z_i / z_{i-1}$ with coordinates

Anton Zeitlin

Space of functions on Z-twisted Miura $(SL(r+1,\hbar)$-opers

$$\updownarrow$$

Space of functions on the intersection of two Lagrangian subvarieties in trigonometric Ruijsenaars-Schneider (tRS) phase space.

$$\text{Bethe equations} \leftrightarrow \{H_k = f_k(\{a_i\})\}$$

Here H_k are tRS Hamiltonians

$$H_k = \sum_{\substack{J\subset\{1,\dots,r+1\}\\ |J|=k}} \prod_{\substack{i\in J\\ j\notin J}} \frac{\xi_i - \hbar\xi_j}{\xi_i - \xi_j} \prod_{m\in J} p_m$$

and f_k are elementary symmetric functions of a_i.

P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

E. Frenkel, P. Koroteev, D. Sage, A.Z. '20

Let us "complete" Miura $(SL(r+1),\hbar)$-opers: $(\overline{GL}(\infty),\hbar)$:

$$A(u)=\prod_{i=+\infty}^{-\infty} g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i},\quad g_i(u)=z_i\frac{Q_+^i(\hbar u)}{Q_+^i(u)}.$$

Infinite-dimensional QQ-system:

$$\xi_{i+1}Q_i^+(\hbar u)Q_i^-(u)-\xi_i Q_i^+(u)Q_i^-(\hbar u)=\Lambda_i(u)Q_{i-1}^+(u)Q_{i+1}^+(\hbar u)\,,\; i=1,\dots,r,$$

where $\xi_i=z_i/z_{i-1}$.

Impose periodic condition: $VA(u)V^{-1}=\xi A(pu)$, where V corresponds to automorphism of Dynkin diagram $i\to i+1$.

V can be actually relized as an "infinite" Coxeter element of standard order.

That corresponds to $Q_j^{\pm}(u)=Q^{\pm}(p^j u)$, $\Lambda_j(u)=\xi^j\Lambda(u)$, $\xi_j=\xi^j$:

$$\xi Q^+(\hbar u)Q^-(u)-Q^+(u)Q^-(\hbar u)=\Lambda(u)Q^+(up^{-1})Q^+(\hbar pu)$$

$\hbar$-Opers for $\widehat{\widehat{\mathfrak{gl}}}(1)$ and Bethe ansatz

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$(\overline{GL}(\infty),\hbar)$:

$$A(u)=\prod_{i=+\infty}^{-\infty} g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i}, \quad g_i(u)=z_i\frac{Q^i_+(\hbar u)}{Q^i_+(u)}.$$

Infinite-dimensional QQ-system:

$$\xi_{i+1}Q_i^+(\hbar u)Q_i^-(u)-\xi_i Q_i^+(u)Q_i^-(\hbar u)=\Lambda_i(u)Q^+_{i-1}(u)Q^+_{i+1}(\hbar u)\,,\, i=1,\dots,r,$$

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Applications

- Quantum/classical duality: duality between Bethe equations and multiparticle systems

  P. Koroteev, D. Sage, A. Z., *(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality*, Comm. Math. Phys., 381 (2021) 641-672, arXiv:1811.09937

- Quantum equivariant K-theory of Nakajima quiver varieties and 3D mirror symmetry

  P. Koroteev, A.Z., *Toroidal q-Opers*, to appear in Journal of the Institute of Mathematics of Jussieu, in press, arXiv:2007.11786

  P. Koroteev, A. Z., *3d Mirror Symmetry for Instanton Moduli Spaces*, arXiv:2105.00588

- Applications to ODE/IM correspondence: affine G-opers and $(G, \hbar)$-opers

  E. Frenkel, P. Koroteev, A.Z., in progress

Applications

- Quantum/classical duality: duality between Bethe equations and multiparticle systems

  P. Koroteev, D. Sage, A. Z., *(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality*, Comm. Math. Phys., 381 (2021) 641-672, arXiv:1811.09937

- Quantum equivariant K-theory of Nakajima quiver varieties and 3D mirror symmetry

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Happy Birthday, Igor!