Quantization of Supersymmetric Korteweg-de Vries Equation and Integrable Structure of Superconformal Field Theory

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1 Classical super-KdV theory

osp(1|2) superalgebra:

$$\begin{aligned} [h, X_\pm] &= \pm 2X_\pm \quad [h, v_\pm] = \pm v_\pm \quad [X_+, X_-] = h \\ [v_\pm, v_\pm]_+ &= \pm 2X_\pm \quad [v_+, v_-]_+ = -h \quad [X_\pm, v_\mp] = v_\pm \\ [X_\pm, v_\pm] &= 0. \end{aligned} \tag{1}$$

Two equivalent $\mathcal{L}$ operators:

$$\begin{aligned} \mathcal{L}_F &= D - D\Psi h - (iv_+\sqrt{\lambda} - \theta\lambda X_-) \\ \mathcal{L}_B &= \partial_u - \phi'(u)h - \sqrt{\lambda/2}\,\xi(u)v_+ - \lambda(X_+ + X_-) \\ D &= \partial_\theta + \theta\partial_u, \end{aligned} \tag{2}$$

$u \in S^1 \times R$, θ **is a Grassmann variable**

$\Psi(u,\theta) = \phi(u) - i\frac{\theta}{\sqrt{2}}\xi(u)$,

Boundary conditions for the fields ϕ and ξ are:

$$\begin{aligned} \phi(u+2\pi) &= \phi(u) + 2\pi i p, \\ \xi(u+2\pi) &= \pm\xi(u) \end{aligned} \tag{3}$$

Poisson brackets:

$$\{\xi(u),\xi(v)\} = -2\delta(u-v) \qquad (4)$$
$$\{\phi(u),\phi(v)\} = \frac{1}{2}\epsilon(u-v)$$

Miura transformation:

$$\begin{aligned} U(u) &= -\phi''(u) - \phi'^2(u) - \frac{1}{2}\xi(u)\xi'(u) \qquad (5)\\ \alpha(u) &= \xi'(u) + \xi(u)\phi'(u)\\ \{\alpha(u),\alpha(v)\} &= 2(\delta''(u-v) + U(u)\delta(u-v))\\ \{U(u),\alpha(v)\} &= 3\alpha(u)\delta'(u-v) + \alpha'(u)\delta(u-v)\\ \{U(u),U(v)\} &= \delta'''(u-v) + 2U'(u)\delta(u-v)\\ &\quad +4U(u)\delta'(u-v) \end{aligned}$$

Integrals of Motion:

$$\begin{aligned} I_1^{(cl)} &= \int U(u)du \qquad (6)\\ I_3^{(cl)} &= \int (U^2(u) + 2\alpha(u)\alpha'(u))du\\ I_5^{(cl)} &= \int ((U')^2(u) - 2U^3(u) + 8\alpha'(u)\alpha''(u) +\\ &\quad +12\alpha'(u)\alpha(u)U(u)du\\ &\quad . \qquad . \qquad .\\ &\quad \{I_{2k-1}^{(cl)}, I_{2l-1}^{(cl)}\} = 0 \end{aligned}$$

$I_3^{(cl)}$ **gives the sKdV equation:**

$$\begin{aligned} \alpha_t &= -4\alpha_{uuu} - 6U\alpha_u - 3U_u\alpha \\ U_t &= -U_{uuu} - 6UU_u - 6\alpha\alpha_{uu} \end{aligned} \tag{7}$$

The linear problem for the $\mathcal{L}_B$ operator is:

$$\pi_s(\mathcal{L}_B)\chi(u) = 0, \tag{8}$$

where π_s is a finite-dimensional irreducible representation of osp(1|2). The solution can be written in such a form:

$$\begin{aligned} \chi(u) &= \pi_s(e^{\phi(u)h}(Pexp\int_0^u du'(\sqrt{\lambda/2}\xi(u')v_+e^{-\phi(u')} \\ &+ \lambda(X_+e^{-2\phi(u')} + X_-e^{2\phi(u')})))\chi_0 \\ &\chi_0 \in C^{2s+1} \end{aligned} \tag{9}$$

The corresponding monodromy matrix is:

$$\begin{aligned} M_s(\lambda) &= \pi_s(e^{2\pi iph}(Pexp\int_0^{2\pi} du'(\sqrt{\lambda/2}\xi(u')v_+e^{-\phi(u')} \\ &+ \lambda(X_+e^{-2\phi(u')} + X_-e^{2\phi(u')}))) \end{aligned} \tag{10}$$

Defining auxiliary matrices:

$$\boldsymbol{\pi_s(L(\lambda)) = L_s(\lambda) = \pi_s(e^{-\pi i p h}) M_s(\lambda)} \tag{11}$$

one obtains the following Poisson bracket relation:

$$\boldsymbol{\{L_s(\lambda) \otimes, L_{s'}(\mu)\} = [r_{ss'}(\lambda\mu^{-1}), L_s(\lambda) \otimes L_{s'}(\mu)]} \tag{12}$$

where $\boldsymbol{r_{ss'}(\lambda\mu^{-1}) = \pi_s \otimes \pi_{s'}(r)}$ is the classical trigonometric osp(1|2) r-matrix.

$$\begin{aligned} \boldsymbol{r(\lambda\mu^{-1})} &= \boldsymbol{\frac{1}{2}\frac{\lambda\mu^{-1} + \lambda^{-1}\mu}{\lambda\mu^{-1} - \lambda^{-1}\mu} h \otimes h} \\ &+ \boldsymbol{\frac{2}{\lambda\mu^{-1} - \lambda^{-1}\mu}(X_+ \otimes X_- + X_- \otimes X_+)} \\ &+ \boldsymbol{\frac{1}{(\lambda\mu^{-1} - \lambda^{-1}\mu)}\left(\sqrt{\frac{\mu}{\lambda}} v_+ \otimes v_- - \sqrt{\frac{\lambda}{\mu}} v_- \otimes v_+\right)} \end{aligned} \tag{13}$$

$$\begin{aligned} &\boldsymbol{T_s(\lambda) = str M_s(\lambda),} \\ &\boldsymbol{\{T_s(\lambda), T_{s'}(\mu)\} = 0} \end{aligned} \tag{14}$$

Expanding log $\boldsymbol{T_1(\lambda)}$ by degrees of $\boldsymbol{\lambda^{-1}}$ we can get that the coefficients in the expansion are local IM.

2 Free field representations of superconformal algebra

$$
\begin{aligned}
&-\beta^2 T(u) =: \phi'^2(u) : +(1-\beta^2/2)\phi''(u) + \frac{1}{2} : \xi\xi'(u) : + \frac{\varepsilon\beta^2}{16} \\
&\frac{i^{1/2}\beta^2}{\sqrt{2}} G(u) = \phi'\xi(u) + (1-\beta^2/2)\xi'(u)
\end{aligned} \tag{15}
$$

is the quantum version of Miura tramsformation.

$$
\phi(u) = iQ + iPu + \sum_n \frac{a_{-n}}{n} e^{inu}; \tag{16}
$$

$$
[Q,P] = \frac{i}{2}\beta^2 \qquad [a_n, a_m] = \frac{\beta^2}{2} n \delta_{n+m,0}
$$

$$
\xi(u) = i^{-1/2} \sum_n \xi_n e^{-inu} \qquad \{\xi_n, \xi_m\} = \beta^2 \delta_{n+m,0}
$$

Two types of boundary conditions on $\xi : \xi(u+2\pi) = \pm\xi(u)$ "+" sign corresponds to the case when ξ is integer modded (the Ramond "R" sector), "–" sign corresponds to the case when ξ is half-integer modded (the Neveu-Schwarz "NS" sector). $\varepsilon = 0$ in the R sector and $\varepsilon = 1$ in the NS sector.
One can expand:

$$
T(u) = \sum_n L_{-n} e^{inu} - \frac{\hat{c}}{16}, \qquad G(u) = \sum_n G_{-n} e^{inu} \tag{17}
$$

where $\hat{c} = 5 - 2(\frac{\beta^2}{2} + \frac{2}{\beta^2})$ and L_n, G_m generate the superconformal algebra.

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n,-m} \\
[L_n, G_m] &= (\frac{n}{2} - m)G_{m+n} \\
[G_n, G_m] &= 2L_{n+m} + \frac{\hat{c}}{2}(n^2 - 1/4)\delta_{n,-m}
\end{aligned} \tag{18}
$$

The classical limit can be obtained by a substitution:

$$
\begin{aligned}
&T(u) = -\frac{\hat{c}}{4}U(u) \qquad G(u) = -\frac{\hat{c}}{2\sqrt{2i}}\alpha(u) \\
&\hat{c} \to -\infty \quad (\beta^2 \to 0) \\
&[,] \to \frac{4\pi}{i\hat{c}}\{,\}
\end{aligned} \tag{19}
$$

The Fock module F_p:

$$
P|p\rangle = p|p\rangle, \quad a_n|p\rangle = 0 \quad \xi_m|p\rangle = 0 \quad n, m > 0 \tag{20}
$$

For the R sector $|p\rangle$ is degenerate:

$$
|p, +\rangle = \xi_0|p, -\rangle \tag{21}
$$

$\boldsymbol{F_p}$ is isomorphic to the superVirasoro module:

$$\begin{aligned} L_0|p\rangle &= \Delta_{NS}|p\rangle, \quad \Delta_{NS} = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}-1}{16} \\ L_0|p,\pm\rangle &= \Delta_R|p,\pm\rangle, \quad \Delta_R = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}}{16}, \\ |p,+\rangle &= \frac{\beta^2}{\sqrt{2}p} G_0|p,-\rangle \end{aligned} \tag{22}$$

The Fock module splits into the direct sum of finite-dimensional subspaces by means of L_0 operator:

$$\begin{aligned} F_p &= \oplus_{k=0}^{\infty} F_p^{(k)}, \\ L_0 F_p^{(k)} &= (\Delta + k) F_p^{(k)} \end{aligned} \tag{23}$$

L_0 is one of the quantum conservation laws, so:

$$I_{2l-1}^{(q)} F_p^{(k)} = I_{2l-1;k}(\Delta) F_p^{(k)} \tag{24}$$

Thus the problem of the diagonalization of local IM reduced to the finite algebraic problem, which, however, rapidly becomes very complex at higher levels.

3 Quantum Monodromy Matrix and Fusion Relations

In the quantum case the underlying algebraic structure is quantum affine superalgebra $\widehat{osp}_q(1|2)$. This algebra is generated by five elements:

$$\boxed{h_d, h_{\pm\alpha}, h_{\pm\alpha_0}, e_{\pm\alpha}, e_{\pm\alpha_0} \qquad (25)}$$

with the following commutation relations:

$$
\boxed{
\begin{aligned}
&[h_\gamma, h'_\gamma] = 0 \quad (\gamma, \gamma' = d, \alpha, \alpha_0) \qquad (26)\\
&[e_\beta, e'_\beta] = \delta_{\beta,\beta'}[h_\beta] \quad (\beta, \beta' = d, \alpha, \alpha_0)\\
&[h_d, e_{\pm\alpha_0}] = \pm e_{\pm\alpha_0}\\
&[h_d, e_{\pm\alpha}] = 0\\
&[h_{\alpha_0}, e_{\pm\alpha_0}] = 2e_{\pm\alpha_0}\\
&[h_{\alpha_0}, e_{\pm\alpha}] = \mp e_{\mp\alpha}\\
&[h_\alpha, e_{\pm\alpha}] = \pm\frac{1}{2}e_{\pm\alpha}\\
&[h_\alpha, e_{\pm\alpha_0}] = \mp e_{\alpha_0}\\
&[[e_{\pm\alpha}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0\\
&[e_{\pm\alpha}, [e_{\pm\alpha}, [e_{\pm\alpha}[e_{\pm\alpha}, [e_{\pm\alpha}, [e_{\pm\alpha_0}]_q]_q]_q]_q]_q = 0
\end{aligned}
}
$$

where $[h_\beta] = \frac{q^{h_\beta} - q^{-h_\beta}}{q - q^{-1}}$ and $[,]_q$ is the super q-commutator: $[e_a, e_b]_q = e_a e_b - q^{(a,b)}(-1)^{\theta(a)\theta(b)} e_b e_a$.

The finite-dimensional irreducible representations have the following explicit form:

$$\begin{aligned}
h_0|j,m\rangle &= 2m|j,m\rangle \\
e_{\alpha_0}|j,m\rangle &= \sqrt{[j-m][j+m+1]}|j,m+1\rangle \\
e_{-\alpha_0}|j,m\rangle &= \sqrt{[j+m][j-m+1]}|j,m-1\rangle \\
e_{\alpha}|j,m\rangle &= (-1)^{-2j}\sqrt{\alpha(j)[j-m+1]}|j+1/2,m-1/2\rangle \\
&+ \sqrt{\alpha(j-1/2)[j+m]}|j-1/2,m-1/2\rangle \\
e_{-\alpha}|j,m\rangle &= -\sqrt{\alpha(j)[j+m+1]}|j+1/2,m+1/2\rangle \\
&- (-1)^{2j}\sqrt{\alpha(j-1/2)[j-m]}|j-1/2,m+1/2\rangle
\end{aligned} \tag{27}$$

where $j = 0, ..., s/2$, $m = -j, ..., j$,

$$\alpha(j) = \frac{[j+1][j+1/2]}{[2j+2][2j+1]} \tag{28}$$

$$\Big((-1)^{s-2j+1}\Big(\frac{q^{s/2+1}}{q^{1/2}+1} + \frac{q^{-s/2-1}}{q^{-1/2}+1}\Big) + \tag{29}$$

$$+\frac{q^{j+1}}{q^{1/2}+1} + \frac{q^{-j-1}}{q^{-1/2}+1}\Big) \tag{30}$$

We denote these representations by $\pi_s^{(q)}$. In the classical limit $q \to 1$ one obtains $\pi_s^{(1)} = \oplus_{k=0}^{[s/2]} \pi_{s-2k}$.

The quantum version of auxiliary operator $\boldsymbol{L_s(\lambda)}$:

$$\boldsymbol{L_s^{(q)}(\lambda) \ = \ \pi_s^{(q)}(e^{-i\pi Ph_0} Pexp(\int_0^{2\pi} du \lambda e_{\alpha_0} : e^{2\phi(u)} : \\ + \ \sqrt{\lambda}\xi(u) : e^{-\phi(u)} : e_\alpha)),} \tag{31}$$

where q=$\boldsymbol{e^{i\pi\beta^2}}$. It should be noted that one term is missing in the P-exponent in comparison with the classical case! Integrals in $\boldsymbol{L_s^{(q)}(\lambda)}$ are convergent for

$$\boldsymbol{-\infty < \hat{c} < 0} \tag{32}$$

and need regularization for a wider region.
$\boldsymbol{L_s^{(q)}(\lambda)}$ satisfy the so-called RTT-relation:

$$\boldsymbol{R_{ss'}(\lambda\mu^{-1})(L_s^{(q)}(\lambda) \otimes I)(I \otimes L_{s'}^{(q)}(\mu)) = \\ (I \otimes L_{s'}^{(q)}(\mu))(L_s^{(q)}(\lambda) \otimes I)R_{ss'}(\lambda\mu^{-1}),} \tag{33}$$

where $\boldsymbol{R_{ss'}(\lambda\mu^{-1})}$ is the trigonometric solution of the corresponding Yang-Baxter equation which acts in the space $\boldsymbol{\pi_s \otimes \pi_{s'}}$.
This is the quantization of the Poisson bracket classical r-matrix relation.

"Transfer matrices":

$$\boldsymbol{T}_s^{(q)}(\boldsymbol{\lambda}) = \mathbf{str}\boldsymbol{\pi}_s(e^{-i\pi p h_0}\boldsymbol{L}_s^{(q)}(\boldsymbol{\lambda})) \tag{34}$$

One obtains that

$$[\boldsymbol{T}_s^{(q)}(\boldsymbol{\lambda}), \boldsymbol{T}_{s'}^{(q)}(\boldsymbol{\mu})] = \mathbf{0} \tag{35}$$

as a consequence of RTT-relation. Using the first few orders of $\boldsymbol{\lambda}$-expansion for the $\boldsymbol{T}_s$ operators, we get the following "fusion" relation :

$$\begin{aligned}&\boldsymbol{T}_s(\boldsymbol{q}^{1/4}\boldsymbol{\lambda})\boldsymbol{T}_s(\boldsymbol{q}^{-1/4}\boldsymbol{\lambda}) = \\ &\boldsymbol{T}_{s+1}(\boldsymbol{q}^{\frac{1}{2\beta^2}}\boldsymbol{\lambda})\boldsymbol{T}_{s-1}(\boldsymbol{q}^{\frac{1}{2\beta^2}}\boldsymbol{\lambda}) + \boldsymbol{T}_s(\boldsymbol{\lambda})\end{aligned} \tag{36}$$

In the case when β^2 is rational (the case of minimal models), fusion relations become a closed system of equations.

4 Relation to the minimal models of CFT

Two possible versions of quantum Miura transformations:

$$-\beta^2 T(u) =: \phi'^2(u) : +(1-\beta^2/2)\phi''(u) + \frac{1}{2} : \xi\xi'(u) : +\frac{\varepsilon\beta^2}{16}$$
$$\frac{i^{1/2}\beta^2}{\sqrt{2}} G(u) = \phi'\xi(u) + (1-\beta^2/2)\xi'(u) \tag{37}$$

$$-\beta^2 T(u) =: \phi'^2(u) : -(1-\beta^2)\phi''(u) + \frac{1}{2} : \xi\xi'(u) : +\frac{\varepsilon\beta^2}{16}$$
$$\frac{i^{1/2}\beta^2}{\sqrt{2}} G(u) = \phi'\xi(u) - (1-\beta^2)\xi'(u) \tag{38}$$

Possible integrable perturbations:

$$\int d\theta \int_0^{2\pi} du e^{-\Phi} \quad \text{and} \quad \int d\theta \int_0^{2\pi} du\theta e^{2\Phi} \tag{39}$$

$\widehat{osp}(1|2)$ Toda Field Theory:

$$S = \frac{1}{\beta^2} \int du d\bar{u} d\theta d\bar{\theta} [D_{u,\theta}\Phi D_{\bar{u},\bar{\theta}}\Phi + e^{-\Phi} + \theta\bar{\theta} e^{2\Phi}] \tag{40}$$

5 Conclusions

1. **The explicit form of finite dimensional irreducible representations of $\widehat{osp}_q(1|2)$ is obtained**
2. **Quantum version of the monodromy matrix and transfer matrices: simultaneous diagonalisation of the integrals of motion**
3. **Fusion Relations for the transfer matrices**
4. **Relation to the perturbed minimal models of CFT is considered**