Quantization of the Integrable Systems

Classical integrable systems we are considering have two related descriptions: the first one gives possibility to express evolution equations via zero curvature condition

$$\partial_t U(\lambda) - \partial_x V(\lambda) + [U(\lambda), V(\lambda)] = 0, \tag{1}$$

where U and V are polynomially depending on λ Lie algebra valued functions of $x \in S^1$, $t \in R$, the second one allows to write it in Hamiltonian form:

$$w^i_t = \{H, w^i\}, \tag{2}$$

where w^i $(i = 1...n)$ are the potentials from the U and V. The zero curvature condition is a consistence condition of the pair of equations:

$$\partial_x \Psi = U(\lambda)\Psi \tag{3}$$
$$\partial_t \Psi = V(\lambda)\Psi \tag{4}$$

It is useful to consider the monodromy matrix for the first of these two equations: $M(\lambda) = Pexp \int_0^{2\pi} U(\lambda)$. It appears that this object satisfy the following Poisson bracket relation:

$$\{M(\lambda), \otimes M(\mu)\} = [r(\lambda\mu^{-1}), M(\lambda) \otimes M(\mu)] \tag{5}$$

where r is a classical r-matrix. Using this, one can obtain that the traces of monodromy matrices are in involution under the Poisson bracket:

$$\boxed{\{trM(\lambda), trM(\mu)\} = 0 \qquad (6)}$$

Expanding $log\ trM(\lambda)$ by λ^{-1}, tending $\lambda \to \infty$ one obtains that coefficients of this expansion are local Integrals of Motion (IM). Thus, by means of the above involutivity property we find that IM are mutually commuting under the Poisson brackets, i.e. integrability condition. It appears that there is a quantization of this systems which preserves integrability condition. We replace the r-matrix relation by the so-called RTT relation:

$$\boxed{R(M^{(q)}(\lambda)\otimes I)(I\otimes M(\mu)^{(q)}) = (I\otimes M(\mu)^{(q)})(M^{(q)}(\lambda)\otimes I)R \qquad (7)}$$

where $M^{(q)}$ is aquantum monodromy matrix, R is a quantum R-matrix, obeying Yang-Baxter equation:

$$\boxed{R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \qquad (8)}$$

Thus we replaced the underlying symmetry based on Lie algebra g by the symmetry generated by quantum Lie algebra g_q with $q = e^{\xi}$. In the quasiclassical limit $\xi \to 0$ or $q \to 1$ R-matrix will have the form:

$$\boldsymbol{R = 1 + \xi(C - r) + O(\xi^2),} \tag{9}$$

where C is a Casimir operator and r is a classical r-matrix. Thus, it can be easily obtained that in this limit RTT relation transforms into the Poisson bracket r-matrix relation mentioned above. RTT relation leads to:

$$\boldsymbol{[trM(\lambda)^{(q)}, trM(\mu)^{(q)}] = 0} \tag{10}$$

which is the integrability condition in the quantum case.

This method of quantization was well applied to the lattice models and gave the possibility to consider them from a point of view of the so-called Quantum Inverse Scattering Method (QISM). Recently Bazhanov, Lukyanov and Zamolodchikov found a way how to apply this method to the continuous models. This allowed to use full analytic power of the theory of integrable lattice models but did not use intermediate lattice regularization and work in the terms of continuous fields from the very beginning. The method was successfully applied to the c<1 Conformal Field Theory (CFT) and then to CFT with the extended conformal symmetry - $\boldsymbol{W_3}$ algebra. We adapt this approach to the supersymmetric extension of CFT - Superconformal Field Theory (SCFT).