Generalized Teichmüller Spaces, Spin Structures and Ptolemy Transformations

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Outline

Introduction

Cast of characters

Coordinates on Super-Teichmüller space

$\mathcal{N} = 2$ Super-Teichmüller theory

Open problems
Introduction

Let $F^g_s \equiv F$ be the Riemann surface of genus $g$ and $s$ punctures. We assume $s > 0$ and $2 - 2g - s < 0$.

Teichmüller space $T(F)$ has many incarnations:

- $\{\text{complex structures on } F\}$/isotopy
- $\{\text{conformal structures on } F\}$/isotopy
- $\{\text{hyperbolic structures on } F\}$/isotopy

Isotopy here stands for diffeomorphisms isotopic to identity.
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Representation-theoretic definition:

\[ T(F) = \text{Hom}'(\pi_1(F), PSL(2, \mathbb{R}))/PSL(2, \mathbb{R}), \]

where \( \text{Hom}' \) stands for Homs such that the group elements corresponding to loops around punctures are parabolic (\(|\text{tr}| = 2\)).

The image \( \Gamma \in PSL(2, \mathbb{R}) \) is a **Fuchsian group**.

By Poincaré uniformization we have \( F = H^+ / \Gamma \), where \( PSL(2, \mathbb{R}) \) acts on the hyperbolic upper-half plane \( H^+ \) as oriented isometries, given by fractional-linear transformations:

\[ z \rightarrow \frac{az + b}{cz + d}. \]

The punctures of \( \tilde{F} = H^+ \) belong to the real line \( \partial H^+ \), which is called **absolute**.
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The primary object of interest in many areas of mathematics is the *moduli space*:

\[ M(F) = T(F)/MC(F). \]

The *mapping class group* \( MC(F) \): a group of the homotopy classes of orientation preserving homeomorphisms.

\( MC(F) \) acts on \( T(F) \) by outer automorphisms of \( \pi_1(F) \).

The goal is to find a system of coordinates on \( T(F) \), so that the action of \( MC(F) \) is realized in the simplest possible way.
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R. Penner's work in the 1980s: a construction of coordinates associated to the ideal triangulation of $F$:

so that one assigns one positive number $\lambda$-length for every edge.

This provides coordinates for the decorated Teichmüller space:

$$\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$$

- Positive parameters correspond to the "renormalized" geodesic lengths ($\lambda = e^{\delta/2}$)

- $\mathbb{R}_+^s$-fiber provides cut-off parameter (determining the size of the horocycle) for every puncture.
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Let $F = F_{g,n}$ be an oriented surface of genus $g$ with $n$ punctures, $n \geq 1$ and $2g - 2 + n > 0$, and $T_{g,n}$ denote the Teichmüller space of hyperbolic structures on $F$ with finite area. Let $\Delta = (c_1, c_2, \ldots, c_D)$ be an ideal triangulation of $F$, where $D = 6g - 6 + 3n$.

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![Diagram of flips](image)

Ptolemy relation: $ef = ac + bd$

In order to obtain coordinates on $T(F)$, one has to consider shear coordinates $z_e = \log\left(\frac{ac}{bd}\right)$, which are subjects to certain linear constraints.
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\[ \begin{array}{c}
\begin{array}{c}
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\begin{array}{c}
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Transformation of coordinates via the triangulation change is therefore governed by Ptolemy relations. This leads to the prominent geometric example of \textit{cluster algebra}, introduced by S. Fomin and A. Zelevinsky in the early 2000s.

Penner’s coordinates can be used for the quantization of $T(F)$ (L. Chekhov, V. Fock, R. Kashaev, late 90s, early 2000s).

Higher Teichmüller spaces: $PSL(2, \mathbb{R})$ is replaced by some real Lie group $G$.

In the case of real reductive groups $G$ the construction of coordinates was given by V. Fock and A. Goncharov (2003) and sparked a lot of applications in various areas of mathematics/mathematical physics.
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String theory: propagating closed strings generate Riemann surfaces:

*Superstrings*, which, according to string theory, are the fundamental objects for the description of our world, carry extra anticommuting parameters $\theta^i$, called *fermions*:

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That leads to generalizations of Teichmüller spaces, relevant for string theory, called $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Teichmüller spaces $ST(F)$, depending on the number of extra fermionic degrees of freedom.

The corresponding supermoduli spaces were intensively studied by various scientists (E. D’Hocker, D. Phong, ...) in the low genus.

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These $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Teichmüller spaces in the terminology of higher Teichmüller theory are related to *supergroups*

$$OSP(1|2), \quad OSP(2|2)$$

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In the late 80s the problem of construction of Penner’s coordinates on $ST(F)$ was introduced on Yu.I. Manin’s Moscow seminar.

The $\mathcal{N} = 1$ case was solved in:

The $\mathcal{N} = 2$ case is solved recently in:
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Further directions of study:

- Cluster algebras with anticommuting variables  
  (first attempt by V. Ovsienko, arXiv:1503.01894)
- Quantization of super-Teichmüller spaces  
  (first attempt by J. Teschner et al., arXiv:1512.02617)
- Application to supermoduli theory and calculation of superstring amplitudes
- Higher super-Teichmüller theory for supergroups of higher rank
i) Superspaces and supermanifolds

Let $\Lambda(K) = \Lambda^0(K) \oplus \Lambda^1(K)$ be an exterior algebra over field $K = \mathbb{R}, \mathbb{C}$ with (in)finitely many generators $1, e_1, e_2, \ldots$, so that

$$a = a^# + \sum_i a_i e_i + \sum_{ij} a_{ij} e_i \wedge e_j + \ldots, \quad \#: \Lambda(K) \to K$$

$a^#$ is referred to as a body of a supernumber.

If $a \in \Lambda^0(K)$, it is called even (bosonic) number.

If $a \in \Lambda^1(K)$, it is called odd (fermionic) number.

Note, that odd numbers anticommute.
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Superspace $\mathbb{K}^{(n|m)}$ is:

$$\mathbb{K}^{(n|m)} = \{(z_1, z_2, \ldots, z_n | \theta_1, \theta_2, \ldots, \theta_m) : z_i \in \Lambda^0(\mathbb{K}), \theta_j \in \Lambda^1(\mathbb{K})\}$$

One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as even and odd coordinates.

Special spaces:
- Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):
  $$H^+ = \{(z | \theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{C}^{(1|N)} | \text{Im } z^# > 0\}$$

- Positive superspace:
  $$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \ldots, z_n | \theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^{(n|m)} | z_i^# > 0, i = 1, \ldots, n\}$$
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ii) **Supergroup $OSp(1|2)$**

**Definition:**

$(2|1) \times (2|1)$ supermatrices $g$, obeying the relation

$$g^{st} J g = J,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the supertranspose $g^{st}$ of $g$ is given by

$$g^{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

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$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{implies} \quad g^\text{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

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Some remarks:

- Lie superalgebra $osp(1|2)$:

  Three even $h, X_\pm$ and two odd $v_\pm$ generators, satisfying the following commutation relations:

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  [h, v_\pm] = \pm v_\pm, \quad [v_\pm, v_\pm] = \mp 2X_\pm, \quad [v_+, v_-] = h.
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- Note, that the body of the supergroup $OSP(1|2)$ is $SL(2, \mathbb{R})$, not $PSL(2, \mathbb{R})$!
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- Note, that the body of the supergroup $OSP(1|2)$ is $SL(2, \mathbb{R})$, not $PSL(2, \mathbb{R})$!
$OSp(1|2)$ acts on $\mathcal{N} = 1$ super half-plane $H^+$, with the absolute $\partial H^+ = \mathbb{R}^{1|1}$ by superconformal fractional-linear transformations:

$$z \rightarrow \frac{az + b}{cz + d} + \eta \frac{\gamma z + \delta}{(cz + d)^2},$$

$$\eta \rightarrow \frac{\gamma z + \delta}{cz + d} + \eta \frac{1 + \frac{1}{2} \delta \gamma}{cz + d}.$$

Factor $H^+/\Gamma$, where $\Gamma$ is a discrete subgroup of $OSp(1|2)$, such that its projection is a Fuchsian group, are called super Riemann surfaces.
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Factor \( H^+/\Gamma \), where \( \Gamma \) is a discrete subgroup of \( \text{OSp}(1|2) \), such that its projection is a Fuchsian group, are called \textit{super Riemann surfaces}.
Alternatively, *super Riemann surface* is a complex $(1|1)$-supermanifold $S$ with everywhere non-integrable odd distribution $\mathcal{D} \in TS$, such that

$$0 \rightarrow \mathcal{D} \rightarrow TS \rightarrow \mathcal{D}^2 \rightarrow 0 \text{ is exact.}$$

There are more general fractional-linear transformations acting on $H^+$. They correspond to $SL(1|2)$ supergroup, and factors $H^+/\Gamma$ give $(1|1)$-supermanifolds which have relation to $\mathcal{N} = 2$ super-Teichmüller theory.
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iii) Ideal triangulations and trivalent fatgraphs

• Ideal triangulation of $F$: triangulation $\Delta$ of $F$ with punctures at the vertices, so that each arc connecting punctures is not homotopic to a point rel punctures.

• Trivalent fatgraph: trivalent graph $\tau$ with cyclic orderings on half-edges about each vertex.

$\tau = \tau(\Delta)$, if the following is true:

1) one fatgraph vertex per triangle

2) one edge of fatgraph intersects one shared edge of triangulation.
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![Fatgraph for $F_1^1$](image1)

![Fatgraph for $F_0^3$](image2)

It is a fun game, sometimes called "Kirby's game", to traverse the boundary components directly on the fatgraph diagram.
iv) \((\mathcal{N} = 1)\) Super-Teichmüller space

From now on let

\[ ST(F) = \text{Hom}'(\pi_1(F), OSp(1|2))/OSp(1|2). \]

Super-Fuchsian representations comprising Hom' are defined to be those whose projections

\[ \pi_1 \rightarrow OSp(1|2) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R}) \]

are Fuchsian group, corresponding to \(F\).

Trivial bundle \(\tilde{ST}(F) = \mathbb{R}_+^s \times ST(F)\) is called the decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space, \(ST(F) (\tilde{ST}(F))\) has \(2^{2g+s-1}\) connected components labeled by spin structures on \(F\).
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iv) $(N = 1)$ Super-Teichmüller space

From now on let

$$ST(F) = \text{Hom}^\prime(\pi_1(F), OSp(1|2))/OSp(1|2).$$

Super-Fuchsian representations comprising $\text{Hom}^\prime$ are defined to be those whose projections

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v) Spin structures

Textbook definition:

Let $M$ be an oriented $n$-dimensional Riemannian manifold, $P_{SO}$ is an orthonormal frame bundle, associated with $TM$. A spin structure is a 2-fold covering map $P \to P_{SO}$, which restricts to $Spin(n) \to SO(n)$ on each fiber.

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This is not really useful for us, since we want to relate it to combinatorial geometric structures on $F$. 
There are several ways to describe spin structures on $F$:

- **D. Johnson (1980):**

  Quadratic forms $q : H_1(F, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$, which are quadratic with respect to the intersection pairing $\cdot : H_1 \otimes H_1 \rightarrow \mathbb{Z}_2$, i.e. $q(a + b) = q(a) + q(b) + a \cdot b$ if $a, b \in H_1$.

- **S. Natanzon:**

  A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \rightarrow SL(2, \mathbb{R})$ of $\rho : \pi_1 \rightarrow PSL_2(\mathbb{R})$. Quadratic form $q$ is computed using the following rules: trace $\tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.
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• D. Cimasoni and N. Reshetikhin (2007):

Combinatorial description of spin structures in terms of the so-called Kasteleyn orientations and dimer configurations on the one-skeleton of a suitable CW decomposition of $F$. They derive a formula for the quadratic form in terms of that combinatorial data.

• We gave a substantial simplification of the combinatorial formulation of spin structures on $F$ (one of the main results of R. Penner, A. Zeitlin, arXiv:1509.06302):

Equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips:

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![Diagram showing the evolution of orientations](image-url)
Coordinates on $S\tilde{T}(F)$

Fix a surface $F = F_g^s$ as above and

- $\tau \subset F$ is some trivalent fatgraph spine
- $\omega$ is an orientation on the edges of $\tau$ whose class in $\mathcal{O}(\tau)$ determines the component $C$ of $S\tilde{T}(F)$

Then there are global affine coordinates on $C$:

- one even coordinate called a $\lambda$-length for each edge
- one odd coordinate called a $\mu$-invariant for each vertex of $\tau$, the latter of which are taken modulo an overall change of sign.

Alternating the sign in one of the fermions corresponds to the reflection on the spin graph.

The above $\lambda$-lengths and $\mu$-invariants establish a real-analytic homeomorphism

$$C \rightarrow \mathbb{R}^{6g-6+3s|4g-4+2s}/\mathbb{Z}_2.$$
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Superflips

When all $a, b, c, d$ are different edges of the triangulations of $F$,

$$
\begin{align*}
\chi &= \frac{ac}{bd} \text{ denotes the cross-ratio, and the evolution of spin graph follows from the construction associated to the spin graph evolution rule.}
\end{align*}
$$
• These coordinates are natural in the sense that if \( \varphi \in MC(F) \) has induced action \( \tilde{\varphi} \) on \( \tilde{\Gamma} \in S\tilde{T}(F) \), then \( \tilde{\varphi}(\tilde{\Gamma}) \) is determined by the orientation and coordinates on edges and vertices of \( \varphi(\tau) \) induced by \( \varphi \) from the orientation \( \omega \), the \( \lambda \)-lengths and \( \mu \)-invariants on \( \tau \).

• There is an even 2-form on \( S\tilde{T}(F) \) which is invariant under super Ptolemy transformations, namely,

\[
\omega = \sum_v d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a - (d \theta)^2,
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where the sum is over all vertices \( v \) of \( \tau \) where the consecutive half edges incident on \( v \) in clockwise order have induced \( \lambda \)-lengths \( a, b, c \) and \( \theta \) is the \( \mu \)-invariant of \( v \).

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Take instead of \( \lambda \)-lengths shear coordinates \( z_e = \log \left( \frac{ac}{bd} \right) \) for every edge \( e \), which are subject to linear relation: the sum of all \( z_e \) adjacent to a given vertex = 0.
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Sketch of construction via hyperbolic supergeometry

XIXth century perspective on hyperbolic (super)geometry:

\(\text{OSp}(1|2)\) acts on super-Minkowski space \(\mathbb{R}^{2,1|2}\) (in the bosonic case \(\text{PSL}(2,\mathbb{R})\) acts on \(\mathbb{R}^{2,1}\)).

If \(A = (x_1, x_2, y, \phi, \theta)\) and \(A' = (x'_1, x'_2, y', \phi', \theta')\) in \(\mathbb{R}^{2,1|2}\), the pairing is:

\[
\langle A, A' \rangle = \frac{1}{2} (x_1 x'_2 + x'_1 x_2) - yy' + \phi \theta' + \phi' \theta.
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Two surfaces of special importance for us are

- Superhyperboloid \(\mathbb{H}\) consisting of points \(A \in \mathbb{R}^{2,1|2}\) satisfying the condition \(\langle A, A \rangle = 1\)

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OSp(1|2) does not act transitively on $L^+$:

The space of orbits is labelled by odd variable up to a sign.

We pick an orbit of the vector $(1, 0, 0, 0, 0, 0)$ and denote it $L_0^+$.

There is an equivariant projection from $L_0^+$ to $\mathbb{R}^{1|1} = \partial H^+$.

**Goal**: Construction of the $\pi_1$-equivariant lift for all the data from the universal cover $\tilde{F}$, associated to its triangulation to $L_0^+$.

Such equivariant lift gives the representation of $\pi_1$ in $OSp(1|2)$. 
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**Goal:** Construction of the $\pi_1$-equivariant lift for all the data from the universal cover $\tilde{F}$, associated to its triangulation to $L^+_0$.

Such equivariant lift gives the representation of $\pi_1$ in $OSp(1|2)$. 
• There is a unique $OSp(1|2)$-invariant of two linearly independent vectors $A, B \in L_0^+$, and it is given by the pairing $\langle A, B \rangle$, the square root of which we will call $\lambda$-length.

Let $\zeta^b \zeta^e \zeta^a$ be a positive triple in $L_0^+$. Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even $r, s, t$, which have positive bodies, and odd $\theta$ so that

$$g \cdot \zeta^e = t(1, 1, 1, \theta, \theta), \quad g \cdot \zeta^b = r(0, 1, 0, 0, 0), \quad g \cdot \zeta^a = s(1, 0, 0, 0, 0).$$

• The moduli space of $OSp(1|2)$-orbits of positive triples in the light cone is given by $(a, b, e, \theta) \in \mathbb{R}^{3|1}_+/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by fermionic reflection.

On the superline $\mathbb{R}^{1|1}$ the parameter $\theta$ is known as the Manin invariant.
Orbits of 2 and 3 points in $L_0^+$

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On the superline $\mathbb{R}^{1|1}$ the parameter $\theta$ is known as *Manin invariant*. 
Orbits of 4 points in $L_0^+$: basic calculation

Suppose points $A$, $B$, $C$ are put in the standard position. The 4th point $D$, so that two new $\lambda$-lengths are $c$, $d$.

![Diagram showing a quadrilateral with vertices A, B, C, D and labels a, b, c, d.]

Fixing the sign of $\theta$, we fix the sign of Manin invariant $\sigma$ in terms of coordinates of $D$.

Important observation: if we turn the picture upside down, then

$$(\theta, \sigma) \rightarrow (\sigma, -\theta)$$
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![Diagram](image)

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The lift of ideal triangulation to super-Minkowski space

Denote:

- $\Delta$ is ideal triangulation of $F$, $\tilde{\Delta}$ is ideal triangulation of the universal cover $\tilde{F}$
- $\Delta_\infty$ ($\tilde{\Delta}_\infty$)-collection of ideal points of $F$ ($\tilde{F}$).

Consider $\Delta$ together with:

- the orientation on the fatgraph $\tau(\Delta)$,
- coordinate system $\tilde{\mathcal{C}}(F, \Delta)$, i.e.
  - positive even coordinate for every edge
  - odd coordinate for every triangle

We call coordinate vectors $\vec{c}$, $\vec{c}'$ equivalent if they are identical up to overall reflection of sign of odd coordinates.

Let $C(F, \Delta) \equiv \tilde{\mathcal{C}}(F, \Delta)/\sim$. This implies that

$$C(F, \Delta) \sim \mathbb{R}^{6g+3s-6|4g+2s-4}/\mathbb{Z}_2$$
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Then there exists a lift for each $\vec{c} \in \ell : \tilde{\Delta}_\infty \to L_0^+$, with the property:

for every quadrilateral $ABCD$, if the arrow is pointing from $\sigma$ to $\theta$ then the lift is given by the picture from the previous slide up to post-composition with the element of $OSp(1|2)$.

The construction of $\ell$ can be done in a recursive way:

Such lift is unique up to post-composition with $OSp(1|2)$ group element and it is $\pi_1$-equivariant. This allows us to construct representation of $\pi_1$ in $OSP(1|2)$, based on the provided data.
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![Diagram of quadrilateral with labeled vertices and arrows]

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Such lift is unique up to post-composition with $OSp(1|2)$ group element and it is $\pi_1$-equivariant. This allows us to construct representation of $\pi_1$ in $OSP(1|2)$, based on the provided data.
Theorem

Fix $F$, $\Delta$, $\tau(\Delta)$ as before. Let $\omega$ be an orientation, corresponding to a specified spin structure $s$ of $F$. Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

$$\ell_\omega : \tilde{\Delta}_\infty \to L_0^+$$

which is uniquely determined up to post-composition by $OSp(1|2)$ under admissibility conditions discussed above, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates.

There is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \to OSp(1|2)$, uniquely determined up to conjugacy by an element of $OSp(1|2)$ such that

1. $\ell$ is $\pi_1$-equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;
2. $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

$$\rho : \pi_1 \xrightarrow{\hat{\rho}} OSp(1|2) \to SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$$

is a Fuchsian representation for $F$;
3. the space of all lifts $\tilde{\rho} : \pi_1 \xrightarrow{\hat{\rho}} OSp(1|2) \to SL(2, \mathbb{R})$ is in one-to-one correspondence with the spin structures $s$ on $F$. 


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Theorem

Fix $F, \Delta, \tau(\Delta)$ as before. Let $\omega$ be an orientation, corresponding to a specified spin structure $s$ of $F$. Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

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The super-Ptolemy transformations

\[ ef = (ac + bd) \left( 1 + \frac{\sigma \theta \sqrt{\chi}}{1 + \chi} \right), \]

\[ \nu = \frac{\sigma + \theta \sqrt{\chi}}{\sqrt{1 + \chi}}, \quad \mu = \frac{\sigma \sqrt{\chi} - \theta}{\sqrt{1 + \chi}} \]

are the consequence of light cone geometry.
The space of all such lifts $\ell_\omega$ coincides with the decorated super-Teichmüller space $\tilde{S^T}(F) = \mathbb{R}_+^s \times ST(F)$.

In order to remove the decoration, one can pass to shear coordinates $z_e = \log \left( \frac{ac}{bd} \right)$.

It is easy to check that the 2-form

$$\omega = \sum_{\Delta} d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a - (d\theta)^2$$

is invariant under the flip transformations. This is a generalization of the formula for Weil-Petersson 2-form.
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\[ \mathcal{N} = 2 \text{ super-Teichmüller theory: prerequisites} \]

\[ \mathcal{N} = 2 \text{ super-Teichmüller space is related to } OSP(2|2) \text{ supergroup of rank 2.} \]

It is more useful to work with its 3 \times 3 incarnation, which is isomorphic to \( \Psi \ltimes SL(1|2)_0 \), where \( \Psi \) is a certain automorphism of the Lie algebra \( sl(1|2) \cong osp(2|2) \).

\( SL(1|2)_0 \) is a supergroup, consisting of supermatrices

\[
\begin{pmatrix}
a & b & \alpha \\
c & d & \beta \\
\gamma & \delta & f
\end{pmatrix}
\]

such that \( f > 0 \) and their Berezinian = 1.

This group acts on the space \( \mathbb{C}^{1|2} \) as superconformal fractional-linear transformations.

As before, \( \mathcal{N} = 2 \) super-Fuchsian groups are the ones whose projections

\[ \pi_1 \to OSP(2|2) \to SL(2, \mathbb{R}) \to PSL(2, \mathbb{R}) \]

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are Fuchsian.
Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2, \mathbb{R})$.

Therefore, the construction of coordinates requires a new notion: $\mathbb{R}_+$-graph connection.

A $G$-graph connection on $\tau$ is the assignment $h_e \in G$ to each oriented edge $e$ of $\tau$ so that $h_{\bar{e}} = h^{-1}_e$ if $\bar{e}$ is the opposite orientation to $e$. Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex $v$ of $\tau$ such that $h'_e = t_v h_e t^{-1}_w$ for each oriented edge $e \in \tau$ with initial point $v$ and terminal point $w$.

The moduli space of flat $G$-connections on $F$ is isomorphic to the space of equivalent $G$-graph connections on $\tau$.

By the way, spin structures can be identified with equivalence classes of $\mathbb{Z}_2$-graph connections.
Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2, \mathbb{R})$.

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A $G$-graph connection on $\tau$ is the assignment $h_e \in G$ to each oriented edge $e$ of $\tau$ so that $h_{\bar{e}} = h_e^{-1}$ if $\bar{e}$ is the opposite orientation to $e$. Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex $v$ of $\tau$ such that $h'_e = t_v h_e t_w^{-1}$ for each oriented edge $e \in \tau$ with initial point $v$ and terminal point $w$.

The moduli space of flat $G$-connections on $F$ is isomorphic to the space of equivalent $G$-graph connections on $\tau$.

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- we assign to each edge of $\Delta$ a positive even coordinate $e$;
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- we assign to each edge $e$ of a triangle of $\Delta$ a positive even coordinate $h_e$, called the ratio, such that if $h_e$ and $h'_e$ are assigned to two triangles sharing the same edge $e$, they satisfy $h_e h'_e = 1$.

The odd coordinates are defined up to overall sign changes $\theta_i \rightarrow -\theta_i$, as well as an overall involution $(\theta_1, \theta_2) \rightarrow (\theta_2, \theta_1)$.

Assignment implies that the ratios $\{h_e\}$ uniquely define an $\mathbb{R}_+\text{-graph connection on } \tau(\Delta)$.

Gauge transformations: if $h_a, h_b, h_e$ are ratios assigned to a triangle $T$ with odd coordinate $(\theta_1, \theta_2)$, then a vertex rescaling at $T$ is the following transformation:

$$(h_a, h_b, h_e, \theta_1, \theta_2) \rightarrow (uh_a, uh_b, uh_e, u^{-1} \theta_1, u \theta_2)$$

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We say that two coordinate vectors of $\tilde{C}(F, \Delta)$ are equivalent if they are related by a finite number of such vertex rescalings (i.e. gauge transformations). In particular the underlying $\mathbb{R}_+-$graph connections on $\tau$ are equivalent.

Let $C(F, \Delta) := \tilde{C}(F, \Delta)/\sim$ be the equivalent classes of coordinate vectors. Then it can be represented by coordinates with $h_a h_b h_e = 1$ for the ratios of the same triangle. This implies that

$$C(F, \Delta) \sim \mathbb{R}^8g+4s−7|8g+4s−8/\mathbb{Z}_2 \times \mathbb{Z}_2$$

Note, that two involutions we have, one corresponding to the fermion reflection and another one corresponding to $\Psi$ give rise to two spin structures, which enumerate components of the $\mathcal{N} = 2$ super-Teichmüller space.

The light cone $L^+_0$ and upper sheet hyperboloid $\mathbb{H}^+_0$ in this case are certain orbits in a pseudo-euclidean superspace $\mathbb{R}^{2,2|4}$. 
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Theorem

Fix $F, \Delta, \tau$ as before. Let $\omega_{\text{sign}} := \omega_{s_{\text{sign}}, \tau}$ be a representative, corresponding to a specified spin structure $s_{\text{sign}}$ of $F$, and let $\omega_{\text{inv}} := \omega_{s_{\text{inv}}, \tau}$ be the representative of another spin structure $s_{\text{inv}}$.

Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$ there exists a map called the lift, 

$$\ell_{\omega_{\text{sign}}, \omega_{\text{inv}}} : \tilde{\Delta}_\infty \to L^+_0,$$

which is uniquely determined up to post-composition by $OSp(2|2)$ under some admissibility conditions, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates. Then there is a representation

$$\hat{\rho} : \pi_1 := \pi_1(F) \to OSp(2|2),$$

uniquely determined up to conjugacy by an element of $OSp(2|2)$ such that

1. $\ell$ is $\pi_1$-equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;

2. $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

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Generic Ptolemy transformations are:

\[
\begin{align*}
\begin{array}{cc}
 a & b \\
 \theta_1, \theta_2 & e \\
 \sigma_1, \sigma_2 & c \\
 d &
\end{array}
\end{align*}
\xrightarrow{h}

\begin{align*}
\begin{array}{cc}
 a & b \\
 f & \\
 \mu_1, \mu_2 & \nu_1, \nu_2 \\
 d & c
\end{array}
\end{align*}
\]
and the transformation formulas are as follows:

\[ ef = (ac + bd) \left( 1 + \frac{h_e^{-1} \sigma_1 \theta_2}{2(\sqrt{\chi} + \sqrt{\chi^{-1}})} + \frac{h_e \sigma_2 \theta_1}{2(\sqrt{\chi} + \sqrt{\chi^{-1}})} \right), \]

\[ \mu_1 = \frac{he \theta_1 + \sqrt{\chi} \sigma_1}{\mathcal{D}}, \quad \mu_2 = \frac{h_e^{-1} \theta_2 + \sqrt{\chi} \sigma_2}{\mathcal{D}}, \]

\[ \nu_1 = \frac{\sigma_1 - \sqrt{\chi} h_e \theta_1}{\mathcal{D}}, \quad \nu_2 = \frac{\sigma_2 - \sqrt{\chi} h_e^{-1} \theta_2}{\mathcal{D}}, \]

\[ h'_a = \frac{h_a}{h_e c_\theta}, \quad h'_b = \frac{h_b c_\theta}{h_e}, \quad h'_c = h_c \frac{c_\theta}{c_\mu}, \quad h'_d = h_d \frac{c_\nu}{c_\theta}, \quad h_f = \frac{c_\sigma}{c_\theta^2}, \]

where

\[ \mathcal{D} := \sqrt{1 + \chi + \frac{\sqrt{\chi}}{2} (h_e^{-1} \sigma_1 \theta_2 + h_e \sigma_2 \theta_1)}, \]

\[ c_\theta := 1 + \frac{\theta_1 \theta_2}{6}. \]
Remarks

The space of all lifts $\ell^{\omega_{\text{sign}},\omega_{\text{inv}}}$ is called decorated $\mathcal{N} = 2$ super-Teichmüller space, which is again $\mathbb{R}^s_+$-bundle over $\mathcal{N} = 2$ super-Teichmüller space.

Removal of the decoration is done using a similar procedure, using shear coordinates.

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Open problems/directions

1) Cluster superalgebras

2) Weil-Petersson-form in $\mathcal{N} = 2$ case

3) Duality between $\mathcal{N} = 2$ super Riemann surfaces and $(1|1)$-supermanifolds

4) Quantization of super-Teichmüller spaces

5) Weil-Petersson volumes

6) Application to supermoduli theory and calculation of superstring amplitudes

7) Higher super-Teichmüller theory for supergroups of higher rank
Thank you!