Geometric wonders of classical and quantum integrable systems

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November 13, 2023

Example of integrable system: harmonic oscillator

2-dimensional phase space with coordinate q and momentum p:

$$\text{Hamiltonian:} \qquad H = \frac{p^2 + \omega^2 q^2}{2},$$

$$\text{Poisson bracket:} \qquad \{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}.$$

Equations of motion:

$$\left.\begin{aligned} \tfrac{dq}{dt} &= \{H, q\} = p \\ \tfrac{dp}{dt} &= \{H, p\} = -\omega^2 q \end{aligned}\right\} \Rightarrow \frac{d^2 q}{dt^2} + \omega^2 q = 0$$

Action-angle variables: polar coordinates in (q, p)-space.

Energy level set: $L_E = \{p^2 + \omega^2 q^2 = 2E\}$ is a circle.

Equations of motion for action-angle variables (H, ϕ):

$$\frac{d\phi}{dt} = \omega, \qquad \frac{dH}{dt} = 0$$

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Classical integrable systems: what are they?

Symplectic $2n$-manifold M: phase space, which has information of coordinates and momenta of a physical system.

Equations of motion:

$$\frac{df}{dt} = \{H, f\}.$$

Integrability: family of conserved quantities: $\{F_i\}_{i=1}^n$:

$$\{F_i, F_j\} = 0, \quad F_1 = H.$$

Liouville-Arnold theorem:

- Compact connected components of $L_c = \{F_i = c_i\}_{i=1}^n$ are diffeomorphic to $\mathbb{T}^n$.
- Existence of action-angle variables $\{I_i\}_{i=1}^n$, $\{\phi^i\}_{i=1}^n$ in the neighborhood of L_c:

$$\frac{d\phi^i}{dt} = \omega^i, \quad \frac{dI_i}{dt} = 0.$$

Finding action/angle variables is a non-trivial problem.

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Explosion of interest in integrable systems: 60s -70s

Integrable soliton equations in (1+1)-dimensions,
e.g. Korteweg-de Vries (KdV) equation:

$$u_t = -u_{xxx} + 6uu_x.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura'67; P. Lax'68;

L. Faddeev, V. Zakharov'71

Lie-theoretic methods through Lax pair formulation:

$$\frac{dL}{dt} = [A, L],$$

where $L = -\partial_x^2 + u(x,t)$ for KdV.

I. Gelfand, L. Dickey'76; V. Drinfeld, V. Sokolov'85

Inverse Scattering Method (ISM):

spectral data of $L \rightarrow$ action-angle variables

At the same time many finite-dimensional multiparticle integrable systems were discovered: Calogero-Moser, Toda, Ruijsennars-Schneider, etc.

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Quantum Integrable models: 70s-80s

Quantum integrability:

$$[H_i, H_j] = 0, \quad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables → simultaneous diagonalization of H_i.

Quantization of (1+1)-models? Put them on the lattice.

Lattice integrable models → new algebraic structures:

R-matrix and Yang-Baxter equation

accompanied with

algebraic Bethe ansatz

lead to the the discovery of Quantum inverse scattering method (QISM) developed by Leningrad School.

That eventually led to the discovery of quantum groups by Drinfeld and Jimbo.

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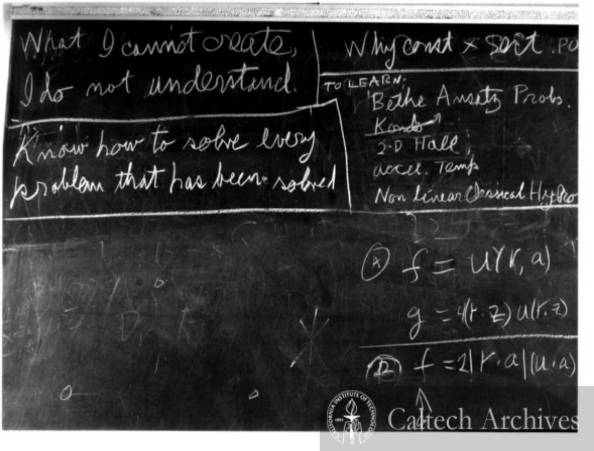

R.P. Feynman: “I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don’t know why. I am trying to understand all this better.”

90s: Geometrization era begins

- Dubrovin, Givental, Kontsevich, Witten established first relations with integrability in the context of enumerative geometry.

  Notable cases:

  - Witten's conjecture, proven by Kontsevich, relating intersection numbers on the moduli space of curves and the τ-function of KdV model.
  - Givental and collaborators: description of the enumerative geometry of flag varieties (quantum cohomology/quantum K-theory) via classical and quantum multiparticle systems of Toda type.

- Feigin, Frenkel, and collaborators, while studying conformal field theory/representation theory of affine Lie algebras, discovered the relation:

  Connections on $\mathbb{P}^1$ called opers $\leftrightarrow$ Gaudin integrable model

  That turned out to be an example of the geometric Langlands correspondence.

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2000s – geometric representation theory and integrable models

- Nakajima, Schiffmann, Varagnolo-Vasserot:

  Geometric realization of representations of quantum groups on cohomology and K-theory of symplectic resolutions, in particular, on Nakajima quiver varieties.

  Okounkov:
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- 2010s: Nekrasov, Shatashvili:

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In this talk

Fusion of:

- Theory of integrable systems
- Geometric representation theory
- Enumerative geometry
- Supersymmetric gauge theories

More concretely, we will discuss the following:

- Nekrasov-Shatashvili conjectures:
  Bethe ansatz solution for quantum integrable systems encodes enumerative invariants of certain symplectic resolutions: quantum cohomology, quantum K-theory.
- On the other hand, geometrization of the relations in the corresponding rings lead to the deformation of the version of geometric Langlands correspondence by Feigin-Frenkel.
- Applications bring together many parts of theoretical physics and mathematics, such as quantum-classical duality, cluster algebras, and 3D mirror symmetry.

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In collaboration with:

P. Koroteev, P. Pushkar, E. Frenkel, D. Sage, A. Smirnov

- P. Koroteev, A.Z., *The Zoo of Opers and Dualities*, arXiv:2208.08031, to appear in Int. Math. Res. Not.
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- P. Koroteev, E. Frenkel, D. Sage, A.Z., *q-Opers, QQ-Systems, and Bethe Ansatz*, to appear in J. Eur. Math. Soc., arXiv:2002.07344
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- P. Koroteev, A.Z., *qKZ/tRS Duality via Quantum K-Theoretic Counts*, Math. Res. Lett.'21
- P. Pushkar, A. Smirnov, A.Z.,*Baxter Q-operator from quantum K-theory*, Adv. Math.'20

Affine algebras and finite-dimensional modules

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules*, form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
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Quantum groups

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$$U_\hbar(\hat{\mathfrak{g}})$$

are deformations of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

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which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:

The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices: FRT construction.

Physical space:

$$\mathcal{H}_{\rm phys} = V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n).$$

Auxiliary spaces: $W(u)$.

Quantum monodromy matrix:

$$M(u) = (Z \otimes \mathrm{Id})\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

Here $\tilde{R}$ is the R-matrix, composed with permutation operator, $Z \in e^{\mathfrak{h}}$ - diagonal.

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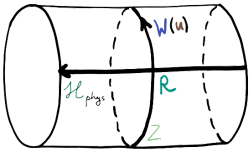
Transfer matrix:



$$T_{W(u)} = \mathrm{Tr}_{W(u)}\Big[M(u)\Big], \quad T_{W(u)} : \mathcal{H}_{\mathrm{phys}} \to \mathcal{H}_{\mathrm{phys}}$$

Integrability:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

follows from Yang-Baxter relation.

Transfer matrices $T_{W(u)}$ generate Bethe algebra:

$$T_{W(u)} = \sum_n u^n I_n, \qquad [I_n, I_m] = 0.$$

Primary goal: diagonalize $\{T_{W(u)}\}$ simultaneously.

$\mathfrak{g} = \mathfrak{sl}(2)$: Heisenberg spin chain

$$\mathcal{H}_{\rm phys} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

$$\text{States}: \quad \uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow$$

Here $\mathbb{C}^2(u)$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic Bethe ansatz as a part of QISM:

$$T_{\mathbb{C}^2(u)} = \mathrm{Tr}_{\mathbb{C}^2(u)}\Big[(Z \otimes \mathrm{Id})\, \tilde{R}_{\mathbb{C}^2(u),\mathcal{H}_{\rm phys}}\Big] = \mathrm{Tr}\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = A(u)+D(u)$$

$$A(u), B(u), C(u), D(u) : \mathcal{H}_{\rm phys} \to \mathcal{H}_{\rm phys}$$

Bethe vectors:

$$|0\rangle = \uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow$$

$$\{B(x_1)\ldots B(x_k)|0\rangle; \quad C(x)|0\rangle = 0\}$$

Commutation relations between A, B, C, D: from Yang-Baxter equation.

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Bethe equations and Q-operator

The eigenvalues are symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{x_i\hbar - x_j}{x_i - x_j\hbar}\,, \quad i = 1, \ldots, k,$$

Special element in the Bethe algebra: Q-operator.

The eigenvalues $\mathcal{Q}(u)$ of the Q-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\mathcal{Q}(u) = \prod_{i=1}^{k}(u - x_i)$$

A real challenge is to describe representation-theoretic meaning of Q-operator for general $\mathfrak{g}$ (possibly infinite-dimensional).

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A real challenge is to describe representation-theoretic meaning of Q-operator for general $\mathfrak{g}$ (possibly infinite-dimensional).

Quantum Knizhnik-Zamolodchikov (Frenkel-Reshetikhin) equations:

$$\Psi(a_1, \dots, qa_k, \dots, a_{i_n}, \{z_i\}) = H_k^{(q)}\Psi(a_1, \dots, , a_n, \{z_i\}),$$
$$+$$
$$\text{commuting } q-\text{difference equations in } Z-\text{components (dynamical)}$$

- Ψ takes values in $\mathcal{H}_{\mathrm{phys}} = V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n)$,
- operators $\{H_i^{(q)}\}$ are expressed in terms of products of R-matrices and twist parameter, e.g.,

$$\Psi(qa_1, \dots, a_n, \{z_i\}) = (Z \otimes 1 \otimes \cdots \otimes 1)R_{V_1,V_n} \dots R_{V_1,V_2}\Psi$$

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Solutions to qKZ

Types of solutions:

- jointly analytic in a chamber of $\{a_i\}$:

  conformal blocks for $U_\hbar(\widehat{\mathfrak{g}})$, i.e. products of Intertwining operators for centrally extended $U_\hbar(\widehat{\mathfrak{g}})$, where central charge is related to q.

- jointly analytic in $\{z_i\}$:

  conformal blocks for deformed W-algebra: $W_{q,t}({}^L\mathfrak{g})$.

The relationship between this solutions is an essential part of

Quantum q-Langlands correspondence

M. Aganagic, E. Frenkel, A. Okounkov'17

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Geometrization: Nakajima quiver varieties

$\mathcal{M}$: affine space constructed via quiver (oriented graph) which is the representation of

$$G = GL(v_1) \times GL(v_2) \cdots \times GL(v_{\mathrm{rank}(\mathfrak{g})}),$$

We build $\mathcal{M}$ as a direct sum of:

- $\oplus_i Hom(V_i, W_i)$, where $dim(V_i) = v_i$, W_i is known as framing
- $\oplus_{i \to j} Hom(V_i, V_j)$

$T^*\mathcal{M}$: phase space with Poisson bracket.

Nakajima quiver variety is a "clever" quotient, called algebraic symplectic reduction:

$$N = T^*\mathcal{M}////G$$

Nakajima, Varagnolo-Vasserot, Maulik-Okounkov: Localized equivariant cohomology/K-theory of N has the structure of weight subspace for representations of $Y_\hbar(\mathfrak{g})/U_\hbar(\mathfrak{g})$. Weight is determined by a collection $v_1, \dots, v_{\mathrm{rank}(\mathfrak{g})}$

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Example: $N_{k,n} = T^*Gr(k,n)$

Let $dim(V) = k$, $dim(W) = n$,

$$\mathcal{M} = Hom(V, W),$$

$$T^*\mathcal{M} = Hom(V, W) \oplus Hom(W, V)$$

The moment map:

$$\mu : T^*\mathcal{M} \to End(V); \quad (A, B) \mapsto BA$$

generates the $GL(V)$-action.

Then

$$T^*Gr(k,n) = N_{k,n} = T^*\mathcal{M}////GL(V) = \mu^{-1}(0)//GL(V) = \mu^{-1}(0)_{ss}/GL(V).$$

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$$\mathcal{V} = T^*\mathcal{M} \times V /\!\!/\!\!/\!\!/ GL(V), \quad \mathcal{W} = T^*\mathcal{M} \times W /\!\!/\!\!/\!\!/ GL(V)$$

For any $\tau \in S(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_k^{\pm 1})$ we have:

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Example: $\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$ corresponds to

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Equivariant K-theory is generated by tautological bundles under tensor product.

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Let $N(n) = \sqcup_k N_{k,n} = \sqcup_k T^* Gr(k,n)$.

Localized K-theory: $K_T(N(n))_{loc}$ as a $\mathbb{Q}(a_1, \dots, a_n, \hbar)$-module is identified with:

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"Classical" Bethe equations: The eigenvalues of the operators of multiplication by τ are $\tau(x_1, \cdots, x_k)$ evaluated at the solutions of the following equations:

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Quasimaps

Quasimap f: $\mathcal{C} = \mathbb{P}^1 \dashrightarrow N_{k,n}$ is the following collection of data:

- vector bundle $\mathscr{V}$ on $\mathbb{P}^1$ of rank k.
- section $f \in H^0(\mathbb{P}^1, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$, satisfying the condition $\mu = 0$, where $\mathscr{M} = Hom(\mathscr{V}, \mathscr{W})$, so that $\mathscr{W}$ is a trivial bundle of rank n.

$$ev_p(f) = f(p) \in [\mu^{-1}(0)/GL(V)] \supset N_{k,n}$$

Quasimap is *stable* if $f(p) \in N_{k,n}$ for all but finitely many points, known as *singularities* of quasimap.

For the moduli space of stable quasimaps

$$QM(N_{k,n})$$

only $\mathscr{V}$ and f vary, while $\mathcal{C}$ and $\mathscr{W}$ remain the same.

$$deg(f) := deg(\mathscr{V}), \quad QM(N_{k,n}) = \sqcup_{d \geq 0} QM^d(N_{k,n}).$$

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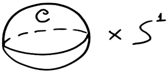
Nekrasov-Shatashvili ideas

In 2009 Nekrasov and Shatashvili looked at 3d SUSY gauge theories on $\mathcal{C} \times S^1$:



Gauge field theory with gauge group $G = \times_{i=1}^{\mathrm{rank}\,\mathfrak{g}} U(v_i)$ defined by a certain action functional $S(\phi_{\{\alpha\}}, A_{\{i\}})$.

- $A_{\{i\}}$: $U(v_i)$-connections (gauge fields)
- $\phi_{\{\alpha\}}$: sections of associated vector $U(v_i)$-bundles, corresponding to the quiver data (matter fields)

Physicists compute path integrals:

$$\langle F \rangle = \int [d\phi_{\{\alpha\}}][dA_{\{i\}}] e^{-S(\phi_{\{\alpha\}}, A_{\{i\}})} F(\phi_{\{\alpha\}}, A_{\{i\}})$$

Minima of S: Moduli of Higgs vacua $\longleftrightarrow$ Nakajima quiver variety:

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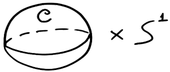

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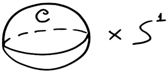

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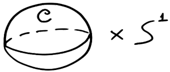

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- $A_{\{i\}}$: $U(v_i)$-connections (gauge fields)
- $\phi_{\{\alpha\}}$: sections of associated vector $U(v_i)$-bundles, corresponding to the quiver data (matter fields)

Physicists compute path integrals:

$$\langle F \rangle = \int [d\phi_{\{\alpha\}}][dA_{\{i\}}] e^{-S(\phi_{\{\alpha\}}, A_{\{i\}})} F(\phi_{\{\alpha\}}, A_{\{i\}})$$

Minima of S: Moduli of Higgs vacua $\longleftrightarrow$ Nakajima quiver variety:

$$T^*\mathcal{M}////G = \mu^{-1}(0)//G = N$$

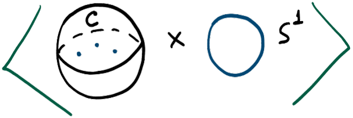
where $\mu = 0$ is a momentum map (low energy configuration) condition.

$\langle F \rangle$ corresponds to weighted K-theoretic counts of quasimaps:

Euler characteristics of equivariant pushforwards of sheaves of quasimap moduli space $QM(N_{k,n})$.

The weight (Kähler) parameter is $Z^{\deg(f)}$, which is exactly twist parameter Z we encountered before.

"Line operators"–traces of holonomy of gauge fields generate quantum K-theory ring, so that the structure constants of algebra are given by:



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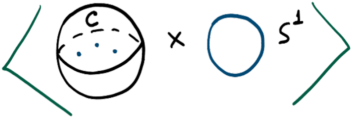

Conjecture of Nekrasov and Shatashvili '09 (through 3D gauge theory):

$$\text{Quantum equivariant K} - \text{theory ring of quiver variety} =$$

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$$\text{Quantum equivariant K}-\text{theory ring of Nakajima variety} =$$

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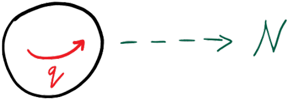
Okounkov'15:

$$\text{q}-\text{difference equations for counting (vertex) functions} =$$
$$\text{qKZ equations} + \text{ dynamical equations}$$

What are we counting?

Quasimaps to quiver varieties:

$$\begin{cases} q, \{a_i\} \to \text{equivariant parameters} \\ \{z_i\} \to \text{"counting" (Kähler) parameters} \end{cases}$$



Anton M. Zeitlin

Quantum tautological classes for $T^*Gr(k,n)$

Deformation of the product: $A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B \, z^d$.

Quantum tautological classes – deformations of $\tau = T^*\mathcal{M} \times \tau(V)$:

$$\hat{\tau}(z) = \tau + \sum_{d=1}^{\infty} \tau_d z^d \in K_T(N(n))[[z]]$$

Theorem.

- The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1, \dots, x_k)$ evaluated at the solutions of Bethe equations:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}, \quad i = 1, \dots, k,$$

- Baxter Q-operator: $Q(u) = \sum_{i=1}^{k} (-1)^i u^{k-i} \left[\Lambda^i V\right](z) \circledast$

P. Pushkar, A. Smirnov, A.Z.,
Baxter Q-operator from quantum K-theory, Adv. Math. '20

Further developments

- Generalization to type A_n, also connection to classical many body-systems for T^*Fl.

  P. Koroteev, P. Pushkar, A. Smirnov, A.Z.,
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QQ-system for T^*Gr

Short exact sequence of bundles:

$$0 \to V \to W \to \hbar \otimes V^{\vee} \to 0$$

$$Q(u) = \sum_{i=1}^{k} (-1)^i u^{k-i} \left[\Lambda^i V\right](z) \circledast$$

$$\widetilde{Q}(u) = \sum_{i=1}^{k} (-1)^i u^{k-i} \left[\Lambda^i V^{\vee}\right](z) \circledast$$

Eigenvalues of these two operators are related via QQ-system:

$$\widetilde{Q}(\hbar u) Q(u) - z Q(\hbar u) \widetilde{Q}(u) = \prod_i (u - a_i)$$

Equivalent to Bethe ansatz equations.

$(G, \hbar)$-connections

$$M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1, \text{ such that } u \to \hbar u.$$

Let $\mathcal{F}_G$ be a G-bundle over $\mathbb{P}^1$:

- G - simple simply connected Lie group associated to Lie algebra $\mathfrak{g}$.
- $\mathcal{F}_G^\hbar$ is a pushforward w.r.t. $M_\hbar$.

$(G, \hbar)$-connection: a meromorphic section of $Hom_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{F}_G, \mathcal{F}_G^\hbar)$.

Locally:

$\hbar$-gauge transformations of $(G, \hbar)$-connection:

$$A(u) \to g(\hbar u)A(u)g^{-1}(u),$$

where $A(u) \in G(u) = G(\mathbb{C}(u))$.

Compare it with standard gauge transformations:

$$\partial_u + A(u) \to g(u)(\partial_u + A(u))g^{-1}(u),$$

where $A(u) \in \mathfrak{g}(u)$.

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Miura $(G, \hbar)$-opers

Miura $(G, \hbar)$-oper is a quadruple : $(\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, such that:

- A is $(G, \hbar)$ - connection
- Oper condition: $\mathcal{F}_{B_+}$: A lies in the Coxeter cell B_+cB_+
- Miura condition: $\mathcal{F}_{B_-}$: preserved by A

$(G, \hbar)$-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(\hbar u) Z v^{-1}(u), \text{ where } Z = \prod_i \zeta_i^{\check{\alpha}_i} \in H, \; v(u) \in G(u).$$

Example: $SL(r+1)$

$$A \in \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ 0 & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * & * \end{bmatrix} \qquad Z - \text{diagonal}$$

Regular singularities: $\{\Lambda_i(u)\}_{i=1}^r$–polynomials on subdiagonal.

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Theorem.
There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura $(G, \hbar)$-opers and the set of nondegenerate polynomial solutions of the QQ-system:

$$\widetilde{\xi}_i Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u) \right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u) \right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

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P. Koroteev, D. Sage, A.Z.,
(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality, Comm. Math. Phys. '21

E. Frenkel, P. Koroteev, D. Sage, A.Z.,
q-Opers, QQ-Systems, and Bethe Ansatz, to appear in J. Eur. Math. Soc., arXiv:2002.07344

In ADE case this QQ-system correspond to the Bethe ansatz equations.
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