

Super-Teichmüller spaces and related structures

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New Trends in Teichmüller Theory and Mapping Class Groups

Oberwolfach

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Outline

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Coordinates on
Super-Teichmüller
space

$\mathcal{N} = 2$
Super-Teichmüller
theory

Open problems

Introduction

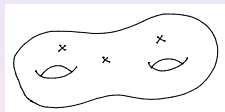
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Let $F_s^g \equiv F$ be the Riemann surface of genus g and s punctures.
We assume $s > 0$ and $2 - 2g - s < 0$.

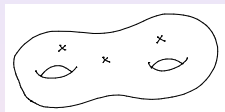


Teichmüller space $T(F)$ has many incarnations:

- ▶ {complex structures on F }/isotopy
- ▶ {conformal structures on F }/isotopy
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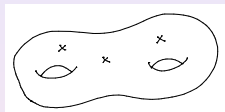


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Representation-theoretic definition:

$$T(F) = \text{Hom}'(\pi_1(F), PSL(2, \mathbb{R})) / PSL(2, \mathbb{R}),$$

where $\rho \in \text{Hom}'$ if

- ▶ ρ is injective
- ▶ identity in $PSL(2, \mathbb{R})$ is not an accumulation point of the image of ρ , i.e. ρ is discrete
- ▶ the group elements corresponding to loops around punctures are parabolic ($|\text{tr}| = 2$)

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The image $\Gamma \in PSL(2, \mathbb{R})$ is a *Fuchsian group*.

By Poincaré uniformization we have $F = H^+/\Gamma$, where $PSL(2, \mathbb{R})$ acts on the hyperbolic upper-half plane H^+ as oriented isometries, given by fractional-linear transformations:

$$z \rightarrow \frac{az + b}{cz + d}.$$

The punctures of $\tilde{F} = H^+$ belong to the real line ∂H^+ , which is called *absolute*.

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The primary object of interest in many areas of mathematics is the *moduli space*:

$$M(F) = T(F)/MC(F).$$

The *mapping class group* $MC(F)$: a group of the homotopy classes of orientation preserving homeomorphisms.

$MC(F)$ acts on $T(F)$ by outer automorphisms of $\pi_1(F)$.

The goal is to find a system of coordinates on $T(F)$, so that the action of $MC(F)$ is realized in the simplest possible way.

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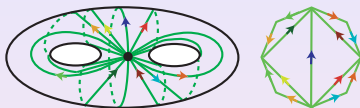
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R. Penner's work in the 1980s: a construction of coordinates associated to the ideal triangulation of F :



so that one assigns one positive number λ -length for every edge.

This provides coordinates for the decorated Teichmüller space:

$$\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$$

- Positive parameters correspond to the "renormalized" geodesic lengths ($\lambda = e^{\delta/2}$)

- \mathbb{R}_+^s -fiber provides cut-off parameter (determining the size of the horocycle) for every puncture.

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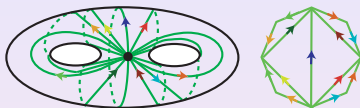
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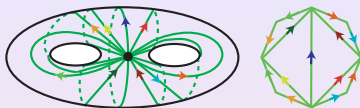
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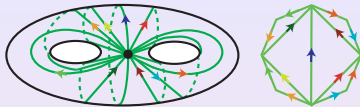
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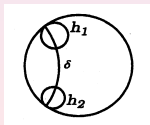


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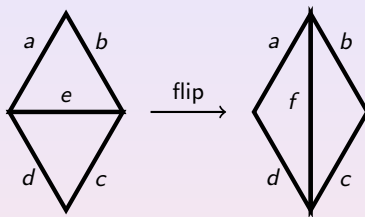
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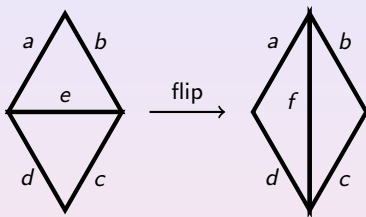
The action of $MC(F)$ can be described combinatorially using elementary transformations called flips:



Ptolemy relation : $ef = ac + bd$

In order to obtain coordinates on $T(F)$, one has to consider *shear coordinates* $z_e = \log\left(\frac{ac}{bd}\right)$, which are subjects to certain linear constraints.

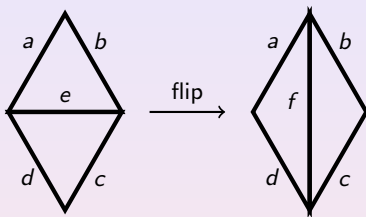
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Transformation of coordinates via the triangulation change is therefore governed by Ptolemy relations. This leads to the prominent geometric example of *cluster algebra*, introduced by [S. Fomin](#) and [A. Zelevinsky](#) in the early 2000s.

Penner's coordinates can be used for the quantization of $T(F)$ ([L. Chekhov](#), [V. Fock](#), [R. Kashaev](#), late 90s, early 2000s).

Higher Teichmüller spaces: $PSL(2, \mathbb{R})$ is replaced by some split semisimple real Lie group G .

In the case of real reductive groups G the construction of coordinates was given by [V. Fock](#) and [A. Goncharov](#) (2003) and sparked a lot of applications in various areas of mathematics/mathematical physics.

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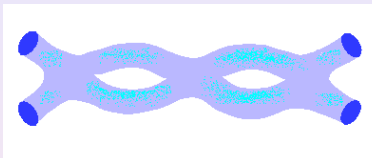
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Superstrings, which, according to string theory, are the fundamental objects for the description of our world, carry extra anticommuting parameters θ^i , called *fermions*:

$$\theta^i \theta^j = -\theta^j \theta^i$$

That can be interpreted as strings propagating along *supermanifolds* called *super Riemann surfaces*.

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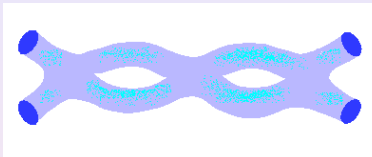
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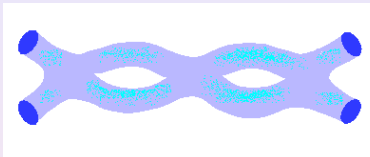
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The corresponding supermoduli spaces were intensively studied by various physicists and mathematicians [L. Crane](#), [J. Rabin](#), [A. Schwarz](#), [A. Voronov](#)..., in particular by [E. D'Hoker](#), [D. Phong](#), in the low genus.

However, not so long ago [R. Donagi](#) and [E. Witten](#) showed that in the higher genus supermoduli spaces are very much involved:

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$$OSP(1|2), \quad OSP(2|2)$$

correspondingly.

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The $\mathcal{N} = 1$ case was solved in:

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i) Superspaces and supermanifolds

Let $\Lambda(\mathbb{K}) = \Lambda^0(\mathbb{K}) \oplus \Lambda^1(\mathbb{K})$ be an exterior algebra over field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with (in)finitely many generators $1, e_1, e_2, \dots$, so that

$$a = a^\# + \sum_i a_i e_i + \sum_{ij} a_{ij} e_i \wedge e_j + \dots, \quad \# : \Lambda(\mathbb{K}) \rightarrow \mathbb{K}$$

$a^\#$ is referred to as a *body* of a supernumber.

If $a \in \Lambda^0(\mathbb{K})$, it is called even (bosonic) number

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Note, that odd numbers anticommute.

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Superspace $\mathbb{K}^{(n|m)}$ is:

$$\mathbb{K}^{(n|m)} = \{(z_1, z_2, \dots, z_n | \theta_1, \theta_2, \dots, \theta_m) : z_i \in \Lambda^0(\mathbb{K}), \theta_j \in \Lambda^1(\mathbb{K})\}$$

One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_j\}$ serve as *even and odd coordinates*.

Special spaces:

- Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):

$$H^+ = \{(z | \theta_1, \theta_2, \dots, \theta_N) \in \mathbb{C}^{(1|N)} \mid \operatorname{Im} z^\# > 0\}$$

- Positive superspace:

$$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \dots, z_n | \theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^{(n|m)} \mid z_i^\# > 0, i = 1, \dots, n\}$$

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$$H^+ = \{(z | \theta_1, \theta_2, \dots, \theta_N) \in \mathbb{C}^{(1|N)} \mid \text{Im } z^\# > 0\}$$

- Positive superspace:

$$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \dots, z_n | \theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^{(n|m)} \mid z_i^\# > 0, i = 1, \dots, n\}$$

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ii) Supergroup $OSp(1|2)$

Definition:

$(1|2) \times (1|2)$ supermatrices g , obeying the relation

$$g^{st} J g = J,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the supertranspose g^{st} of g is given by

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{implies} \quad g^{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

We want a connected component of identity, so we assume that Berezinian (super-analogue of determinant) = 1.

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Some remarks:

- Lie superalgebra $osp(1|2)$:

Three even h, X_{\pm} and two odd v_{\pm} generators, satisfying the following commutation relations:

$$[h, v_{\pm}] = \pm v_{\pm}, \quad [v_{\pm}, v_{\pm}] = \mp 2X_{\pm}, \quad [v_+, v_-] = h.$$

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$OSp(1|2)$ acts on $\mathcal{N} = 1$ super half-plane H^+ , with the absolute $\partial H^+ = \mathbb{R}^{1|1}$ by superconformal fractional-linear transformations:

$$z \rightarrow \frac{az + b}{cz + d} + \eta \frac{\gamma z + \delta}{(cz + d)^2},$$
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iii) ($\mathcal{N} = 1$) Super-Teichmüller space

From now on let

$$ST(F) = \text{Hom}'(\pi_1(F), OSp(1|2))/OSp(1|2).$$

Super-Fuchsian representations comprising Hom' are defined to be those whose projections

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are Fuchsian groups, corresponding to F .

Trivial bundle $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$ is called the decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space, $ST(F)$ ($S\tilde{T}(F)$) has 2^{2g+s-1} connected components labeled by spin structures on F .

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iv) Ideal triangulations and trivalent fatgraphs

- Ideal triangulation of F : triangulation Δ of F with punctures at the vertices, so that each arc connecting punctures is not homotopic to a point rel punctures.
- Trivalent fatgraph: trivalent graph τ with cyclic orderings on half-edges about each vertex.

$\tau = \tau(\Delta)$, if the following is true:

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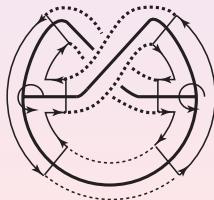
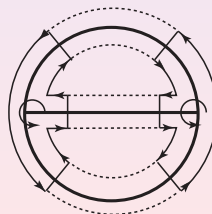
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Fatgraph for F_1^1 Fatgraph for F_0^3

v) Spin structures

Textbook definition:

Let M be an oriented n -dimensional Riemannian manifold, P_{SO} is an orthonormal frame bundle, associated with TM . A *spin structure* is a 2-fold covering map $P \rightarrow P_{SO}$, which restricts to $Spin(n) \rightarrow SO(n)$ on each fiber.

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There are several ways to describe spin structures on F :

- D. Johnson (1980):

Quadratic forms $q : H_1(F, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$, which are quadratic with respect to the intersection pairing $\cdot : H_1 \otimes H_1 \rightarrow \mathbb{Z}_2$, i.e.

$$q(a + b) = q(a) + q(b) + a \cdot b \text{ if } a, b \in H_1.$$

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A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \rightarrow SL(2, \mathbb{R})$ of $\rho : \pi_1 \rightarrow PSL_2(\mathbb{R})$. Quadratic form q is computed using the following rules: $\text{trace } \tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.

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- [D. Cimasoni and N. Reshetikhin \(2007\)](#):

Combinatorial description of spin structures in terms of the so-called Kasteleyn orientations and dimer configurations on the one-skeleton of a suitable CW decomposition of F . They derive a formula for the quadratic form in terms of that combinatorial data.

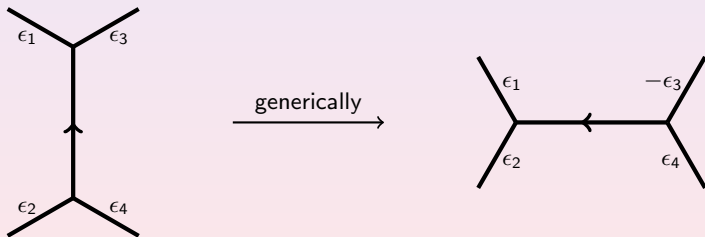
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Equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips:



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Fix a surface $F = F_g^s$ as above and

- ▶ $\tau \subset F$ is some trivalent fatgraph spine
- ▶ ω is an orientation on the edges of τ whose class in $\mathcal{O}(\tau)$ determines the component C of $S\tilde{T}(F)$

Then there are global affine coordinates on C :

- ▶ one even coordinate called a λ -length for each edge
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the latter of which are taken modulo an overall change of sign.

Alternating the sign in one of the fermions corresponds to the reflection on the spin graph.

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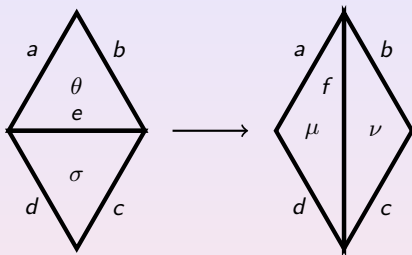
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When all a, b, c, d are different edges of the triangulations of F ,



Ptolemy transformations are as follows:

$$ef = (ac + bd) \left(1 + \frac{\sigma\theta\sqrt{\chi}}{1 + \chi} \right),$$

$$\nu = \frac{\sigma + \theta\sqrt{\chi}}{\sqrt{1 + \chi}}, \quad \mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1 + \chi}}.$$

$\chi = \frac{ac}{bd}$ denotes the cross-ratio, and the evolution of spin graph follows from the construction associated to the spin graph evolution rule.

- These coordinates are natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{\Gamma} \in ST(F)$, then $\tilde{\varphi}(\tilde{\Gamma})$ is determined by the orientation and coordinates on edges and vertices of $\varphi(\tau)$ induced by φ from the orientation ω , the λ -lengths and μ -invariants on τ .

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where the sum is over all vertices v of τ where the consecutive half edges incident on v in clockwise order have induced λ -lengths a, b, c and θ is the μ -invariant of v .

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Sketch of construction via hyperbolic supergeometry

XIXth century perspective on hyperbolic (super)geometry:

$OSp(1|2)$ acts on super-Minkowski space $\mathbb{R}^{2,1|2}$ (in the bosonic case $PSL(2, \mathbb{R})$ acts on $\mathbb{R}^{2,1}$).

If $A = (x_1, x_2, y, \phi, \theta)$ and $A' = (x'_1, x'_2, y', \phi', \theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A' \rangle = \frac{1}{2}(x_1 x'_2 + x'_1 x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- ▶ Superhyperboloid \mathbb{H} consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$
- ▶ Positive super light cone L^+ consisting of points $B \in \mathbb{R}^{2,1|2}$ satisfying $\langle B, B \rangle = 0$,

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If $A = (x_1, x_2, y, \phi, \theta)$ and $A' = (x'_1, x'_2, y', \phi', \theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A' \rangle = \frac{1}{2}(x_1 x'_2 + x'_1 x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- ▶ Superhyperboloid \mathbb{H} consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$
- ▶ Positive super light cone L^+ consisting of points $B \in \mathbb{R}^{2,1|2}$ satisfying $\langle B, B \rangle = 0$,

where $x_1^\#, x_2^\# \geq 0$.

There is an equivariant projection from \mathbb{H} on the $\mathcal{N} = 1$ super upper half-plane H^+ .

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XIXth century perspective on hyperbolic (super)geometry:

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$OSp(1|2)$ does not act transitively on L^+ :

The space of orbits is labelled by odd variable up to a sign.

We pick an orbit of the vector $(1, 0, 0, 0, 0)$ and denote it L_0^+ .

There is an equivariant projection from L_0^+ to $\mathbb{R}^{1|1} = \partial H^+$.

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- There is a unique $OSp(1|2)$ -invariant of two linearly independent vectors $A, B \in L_0^+$, and it is given by the pairing $\langle A, B \rangle$, the square root of which we will call λ -length.

Let $\zeta^b \zeta^e \zeta^a$ be a positive triple in L_0^+ . Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even r, s, t , which have positive bodies, and odd θ so that

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- The moduli space of $OSp(1|2)$ -orbits of positive triples in the light cone is given by $(a, b, e, \theta) \in \mathbb{R}_+^{3|1} / \mathbb{Z}_2$, where \mathbb{Z}_2 acts by fermionic reflection.

On the superline $\mathbb{R}^{1|1}$ the parameter θ is known as *Manin invariant*.

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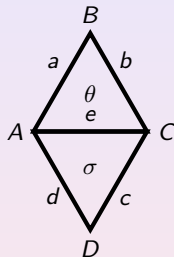
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Orbits of 4 points in L_0^+ : basic calculation

Suppose points A, B, C are put in the standard position.

The 4th point D , so that two new λ - lengths are c, d .



Fixing the sign of θ , we fix the sign of Manin invariant σ in terms of coordinates of D .

Important observation: if we turn the picture upside down, then

$$(\theta, \sigma) \rightarrow (\sigma, -\theta)$$

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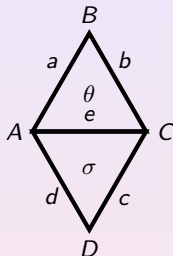
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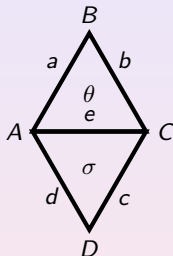
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The lift of ideal triangulation to super-Minkowski space

Denote:

- ▶ Δ is ideal triangulation of F , $\tilde{\Delta}$ is ideal triangulation of the universal cover \tilde{F}
- ▶ Δ_∞ ($\tilde{\Delta}_\infty$)-collection of ideal points of F (\tilde{F}).

Consider Δ together with:

- the orientation on the fatgraph $\tau(\Delta)$,
- coordinate system $\tilde{C}(F, \Delta)$, i.e.
- positive even coordinate for every edge
- odd coordinate for every triangle

We call coordinate vectors \vec{c}, \vec{c}' equivalent if they are identical up to overall reflection of sign of odd coordinates.

Let $C(F, \Delta) \equiv \tilde{C}(F, \Delta) / \sim$. This implies that

$$C(F, \Delta) \simeq \mathbb{R}_+^{6g+3s-6} |4g+2s-4| / \mathbb{Z}_2$$

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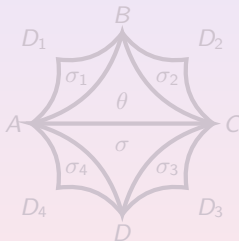
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for every quadrilateral $ABCD$, if the arrow is pointing from σ to θ then the lift is given by the picture from the previous slide up to post-composition with the element of $OSP(1|2)$.

The construction of ℓ can be done in a recursive way:



Such lift is unique up to post-composition with $OSP(1|2)$ group element and it is π_1 -equivariant. This allows us to construct representation of π_1 in $OSP(1|2)$, based on the provided data.

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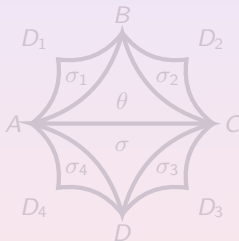
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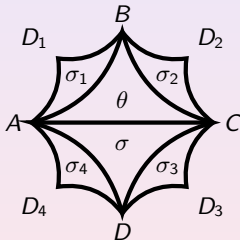
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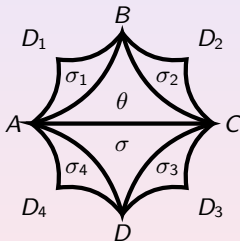


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Theorem

Fix $F, \Delta, \tau(\Delta)$ as before. Let ω be an orientation, corresponding to a specified spin structure s of F . Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

$$\ell_\omega : \tilde{\Delta}_\infty \rightarrow L_0^+$$

which is uniquely determined up to post-composition by $OSp(1|2)$ under admissibility conditions discussed above, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates.

There is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \rightarrow OSp(1|2)$, uniquely determined up to conjugacy by an element of $OSp(1|2)$ such that

- (1) ℓ is π_1 -equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;
- (2) $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

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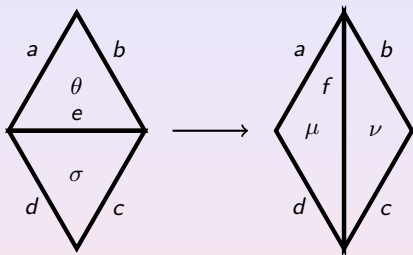
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The super-Ptolemy transformations



$$ef = (ac + bd) \left(1 + \frac{\sigma \theta \sqrt{\chi}}{1 + \chi} \right),$$

$$\nu = \frac{\sigma + \theta \sqrt{\chi}}{\sqrt{1 + \chi}}, \quad \mu = \frac{\sigma \sqrt{\chi} - \theta}{\sqrt{1 + \chi}}$$

are the consequence of light cone geometry.

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The space of all such lifts ℓ_ω coincides with the decorated super-Teichmüller space $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$.

In order to remove the decoration, one can pass to shear coordinates $z_e = \log \left(\frac{ac}{bd} \right)$.

It is easy to check that the 2-form

$$\omega = \sum_{\Delta} d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a - (d\theta)^2$$

is invariant under the flip transformations. This is a generalization of the formula for Weil-Petersson 2-form.

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$$\omega = \sum_{\Delta} d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a - (d\theta)^2$$

is invariant under the flip transformations. This is a generalization of the formula for Weil-Petersson 2-form.

The space of all such lifts ℓ_ω coincides with the decorated super-Teichmüller space $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$.

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Further reduction of the decoration: $S\tilde{T}(F) = \mathbb{R}_+^{6g+3s-6|4g+2s-4} / \mathbb{Z}_2$ is actually an $\mathbb{R}_+^{(s|n_R)}$ decoration over physically relevant Teichmüller space.

Here n_R is the number of Ramond punctures, which means that the small contour γ surrounding the puncture is such that $q[\gamma] = 1$, i.e. $\text{tr}(\tilde{\rho}(\gamma)) > 0$.

On the level of hyperbolic geometry, the appropriate constraint is that the monodromy group element has to be true parabolic, i.e. to be conjugated to the parabolic element of $SL(2, \mathbb{R})$ subgroup.

We formulated it in terms of invariant constraints on shear coordinates in:

I. Ip, R. Penner, A. Zeitlin, [arXiv:1709.06207](https://arxiv.org/abs/1709.06207)

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$\mathcal{N} = 2$ super-Teichmüller theory: prerequisites

$\mathcal{N} = 2$ super-Teichmüller space is related to $OSP(2|2)$ supergroup of rank 2.

It is more useful to work with its 3×3 incarnation, which is isomorphic to $\Psi \ltimes SL(1|2)_0$, where Ψ is a certain automorphism of the Lie algebra $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$.

$SL(1|2)_0$ is a supergroup, consisting of supermatrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix}$$

such that $f > 0$ and their Berezinian = 1.

This group acts on the space $\mathbb{C}^{1|2}$ as superconformal fractional-linear transformations.

As before, $\mathcal{N} = 2$ super-Fuchsian groups are the ones whose projections

$$\pi_1 \rightarrow OSP(2|2) \rightarrow GL^+(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$$

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Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2, \mathbb{R})$.

Therefore, the construction of coordinates requires a new notion:
 \mathbb{R}_+ -graph connection.

A G -graph connection on τ is the assignment $h_e \in G$ to each oriented edge e of τ so that $h_{\bar{e}} = h_e^{-1}$ if \bar{e} is the opposite orientation to e .

Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex v of τ such that $h'_e = t_v h_e t_w^{-1}$ for each oriented edge $e \in \tau$ with initial point v and terminal point w .

The moduli space of flat G -connections on F is isomorphic to the space of equivalent G -graph connections on τ .

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The data giving the coordinate system $\tilde{C}(F, \Delta)$ is as follows:

- ▶ we assign to each edge of Δ a positive even coordinate e ;
- ▶ we assign to each triangle of Δ two odd coordinates (θ_1, θ_2) ;
- ▶ we assign to each edge e of a triangle of Δ a positive even coordinate h_e , called the *ratio*, such that if h_e and h'_e are assigned to two triangles sharing the same edge e , they satisfy $h_e h'_e = 1$.

The odd coordinates are defined up to overall sign changes $\theta_i \rightarrow -\theta_i$, as well as an overall involution $(\theta_1, \theta_2) \rightarrow (\theta_2, \theta_1)$.

Assignment implies that the ratios $\{h_e\}$ uniquely define an \mathbb{R}_+ -graph connection on $\tau(\Delta)$.

Gauge transformations: if h_a, h_b, h_e are ratios assigned to a triangle T with odd coordinate (θ_1, θ_2) , then a *vertex rescaling at T* is the following transformation:

$$(h_a, h_b, h_e, \theta_1, \theta_2) \rightarrow (uh_a, uh_b, uh_e, u^{-1}\theta_1, u\theta_2)$$

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We say that two coordinate vectors of $\tilde{C}(F, \Delta)$ are equivalent if they are related by a finite number of such vertex rescalings (i.e. gauge transformations). In particular the underlying \mathbb{R}_+ -graph connections on τ are equivalent.

Let $C(F, \Delta) := \tilde{C}(F, \Delta) / \sim$ be the equivalent classes of coordinate vectors. Then it can be represented by coordinates with $h_a h_b h_e = 1$ for the ratios of the same triangle. This implies that

$$C(F, \Delta) \simeq \mathbb{R}_+^{8g+4s-7} / \mathbb{Z}_2 \times \mathbb{Z}_2$$

Note, that two involutions we have, one corresponding to the fermion reflection and another one corresponding to Ψ give rise to two spin structures, which enumerate components of the $\mathcal{N} = 2$ super-Teichmüller space.

The light cone L_0^+ and upper sheet hyperboloid \mathbb{H}_0^+ in this case are certain orbits in a pseudo-euclidean superspace $\mathbb{R}^{2,2|4}$.

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$$C(F, \Delta) \simeq \mathbb{R}_+^{8g+4s-7} / \mathbb{Z}_2 \times \mathbb{Z}_2$$

Note, that two involutions we have, one corresponding to the fermion reflection and another one corresponding to Ψ give rise to two spin structures, which enumerate components of the $\mathcal{N} = 2$ super-Teichmüller space.

The light cone L_0^+ and upper sheet hyperboloid \mathbb{H}_0^+ in this case are certain orbits in a pseudo-euclidean superspace $\mathbb{R}^{2,2|4}$.

Theorem

Fix F, Δ, τ as before. Let $\omega_{sign} := \omega_{s_{sign}, \tau}$ be a representative, corresponding to a specified spin structure s_{sign} of F , and let $\omega_{inv} := \omega_{s_{inv}, \tau}$ be the representative of another spin structure s_{inv} .

Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$ there exists a map called the lift,

$$\ell_{\omega_{sign}, \omega_{inv}} : \tilde{\Delta}_{\infty} \rightarrow L_0^+,$$

which is uniquely determined up to post-composition by $OSp(2|2)$ under some admissibility conditions, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates. Then there is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \rightarrow OSp(2|2)$, uniquely determined up to conjugacy by an element of $OSp(2|2)$ such that

- (1) ℓ is π_1 -equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_{\infty}$;
- (2) $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

$$\rho : \pi_1 \xrightarrow{\hat{\rho}} OSp(2|2) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$$

is a Fuchsian representation;

- (3) the lift $\tilde{\rho} : \pi_1 \xrightarrow{\hat{\rho}} OSp(2|2) \rightarrow SL(2, \mathbb{R})$ of ρ does not depend on ω_{inv} , and the space of all such lifts is in one-to-one correspondence with the spin structures ω_{sign} .

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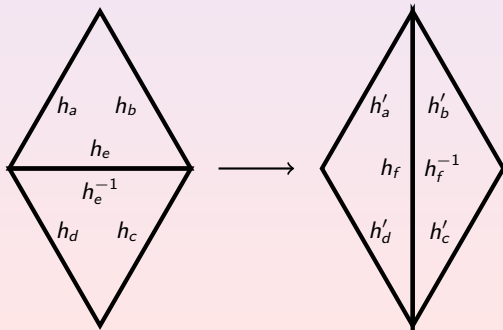
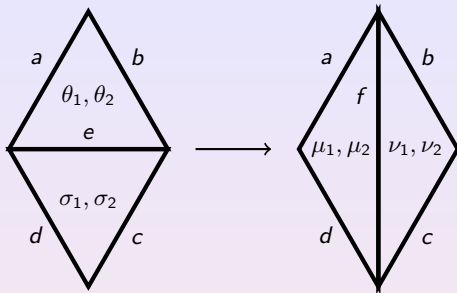
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Generic Ptolemy transformations are:



and the transformation formulas are as follows:

$$ef = (ac + bd) \left(1 + \frac{h_e^{-1}\sigma_1\theta_2}{2(\sqrt{\chi} + \sqrt{\chi^{-1}})} + \frac{h_e\sigma_2\theta_1}{2(\sqrt{\chi} + \sqrt{\chi^{-1}})} \right),$$

$$\mu_1 = \frac{h_e\theta_1 + \sqrt{\chi}\sigma_1}{\mathcal{D}}, \quad \mu_2 = \frac{h_e^{-1}\theta_2 + \sqrt{\chi}\sigma_2}{\mathcal{D}},$$

$$\nu_1 = \frac{\sigma_1 - \sqrt{\chi}h_e\theta_1}{\mathcal{D}}, \quad \nu_2 = \frac{\sigma_2 - \sqrt{\chi}h_e^{-1}\theta_2}{\mathcal{D}},$$

$$h'_a = \frac{h_a}{h_e c_\theta}, \quad h'_b = \frac{h_b c_\theta}{h_e}, \quad h'_c = h_c \frac{c_\theta}{c_\mu}, \quad h'_d = h_d \frac{c_\nu}{c_\theta}, \quad h_f = \frac{c_\sigma}{c_\theta^2},$$

where

$$\mathcal{D} := \sqrt{1 + \chi + \frac{\sqrt{\chi}}{2}(h_e^{-1}\sigma_1\theta_2 + h_e\sigma_2\theta_1)},$$

$$c_\theta := 1 + \frac{\theta_1\theta_2}{6}.$$

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The space of all lifts $\ell_{\omega_{sign}, \omega_{inv}}$ is called decorated $\mathcal{N} = 2$ super-Teichmüller space, which is again \mathbb{R}_+^s -bundle over $\mathcal{N} = 2$ super-Teichmüller space.

Removal of the decoration is done using a similar procedure, using shear coordinates.

The search for the formula of the analogue of Weil-Petersson form is under way. Complication: \mathbb{R}_+ - graph connection provides boson-fermion mixing.

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- 1) Cluster superalgebras
- 2) Weil-Petersson-form in $\mathcal{N} = 2$ case
- 3) Quantization of super-Teichmüller spaces
- 4) Weil-Petersson volumes
- 5) Relation to Strebel theory
- 6) Higher super-Teichmüller theory for supergroups of higher rank

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Thank you!