The geometric meaning of Bethe equations

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Some history of quantum integrable systems and Bethe ansatz

Exactly solvable models of statistical physics: spin chains, vertex models

1930s: Hans Bethe: Bethe ansatz solution of Heisenberg model

1960-70s: R.J. Baxter, C.N. Yang: Yang-Baxter equation, Baxter operator

1980s: Development of "QISM" by Leningrad school leading to the discovery of quantum groups by Drinfeld and Jimbo

Since 1990s: textbook subject and an established area of mathematics and physics.

Geometric interpretations

- Enumerative geometry: quantum K-theory

  Generalization of quantum cohomology in the early 2000s by A. Givental, Y.P. Lee and collaborators. Recently big progress in this direction by A. Okounkov and his school.

  P.Pushkar, A. Smirnov, A.Z., *Baxter Q-operator from quantum K-theory*, arXiv:1612.08723

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- Multiplicative connections, q-opers

  q-deformed version of the classic example of geometric Langlands correspondence, studied in detail by B. Feigin, E. Frenkel, N. Reshetkhin: correspondence between opers (certain connections with regular singularities) and Gaudin models.

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Outline

Loop algebras and evaluation modules

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules* form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
- a_i are values for t

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Quantum group

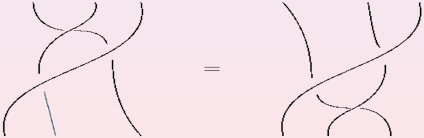
$$U_{\hbar}(\hat{\mathfrak{g}})$$

is a deformation of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1)\otimes V_2(a_2)$$

$$V_2(a_2)\otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:



The generators of $U_{\hbar}(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices (the so-called FRT construction).

Integrability and Baxter algebra

Source of integrability: commuting *transfer matrices*, generating *Baxter algebra* which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

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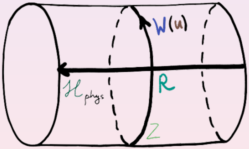
over auxiliary $W(u)$ space:

$$T_W(u) = \mathrm{Tr}_{W(u)}\Big((Z \otimes 1)\ \tilde{R}_{W(u),\mathcal{H}_{phys}} \Big)$$



Here $Z \in e^{\mathfrak{h}}$, where $\mathfrak{h} \in \mathfrak{g}$ are diagonal matrices.

Integrability:

$$[T_{W'}(u'), T_W(u)] = 0$$

There are special transfer matrices is called *Baxter Q-operators*. Such operators generate all Baxter algebra.

Primary goal for physicists is to diagonalize $\{T_W(u)\}$ simultaneously.

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$\mathfrak{g} = \mathfrak{sl}(2)$: XXZ spin chain

Textbook example (and main example in this talk) is XXZ Heisenberg spin chain:

$$\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

States:

$$\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow$$

Here $\mathbb{C}^2$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic method to diagonalize transfer matrices:

Algebraic Bethe ansatz

as a part of Quantum Inverse Scattering Method developed in the 1980s.

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Bethe equations and Q-operator

The eigenvalues are generated by symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}\,, \quad i = 1 \cdots k,$$

so that the eigenvalues $\Lambda(u)$ of the Q-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\Lambda(u) = \prod_{i=1}^{k} (1 + u \cdot x_i)$$

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q-difference equation

Modern way of looking at Bethe ansatz: solving q-difference equations for

$$\Psi(z_1, \dots, z_k; a_1, \dots, a_n) \in V_1(a_1) \otimes \dots \otimes V_n(a_n)[[z_1, \dots, z_k]]$$

known as

Quantum Knizhnik-Zamolodchikov (aka Frenkel-Reshetikhin) equations:

$$\Psi(qa_1, \dots, a_n, \{z_i\}) = (Z \otimes 1 \otimes \dots \otimes 1) R_{V_1,V_n} \dots R_{V_1,V_2} \Psi$$

$$+$$

$$\text{commuting difference equations in } z - \text{variables}$$

Here $\{z_i\}$ are the components of twist variable Z.

The latter series of equations are known as dynamical equations, studied by Etingof, Felder, Tarasov, Varchenko, ...

In $q \to 1$ limit we arrive to an eigenvalue problem. Studying the asymptotics of the corresponding solutions we arrive to Bethe equations and eigenvectors.

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QQ-systems

Another modern view on Bethe ansatz is due to D. Hernandez and E. Frenkel, following earlier papers by V. Bazhanov, S. Lukyanov and A. Zamolodchikov.

Extension of the category of representations of $U_\hbar(\hat{\mathfrak{g}})$ by representations of Borel subalgebra give rise to the so-called QQ-systems.

In the case of $U_\hbar(\hat{\mathfrak{sl}(2)})$ the QQ-system is:

$$z\widetilde{Q}(\hbar u)Q(u) - z^{-1}Q(\hbar u)\widetilde{Q}(u) = \prod_i (u - a_i)$$

Here $Q(u)$ can be viewed as an eigenvalue of the Q-operator.

Key ideas: enumerative geometry of Nakajima varieties

Nekrasov and Shatashvili:

$$\mathrm{Quantum\ equivariant\ K-theory\ ring\ of\ Nakajima\ variety} =$$

$$\mathrm{symmetric\ polynomials\ in\ } x_{i_j} \ / \ \mathrm{Bethe\ equations}$$

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Nekrasov and Shatashvili:

$$\text{Quantum K} - \text{theory ring of Nakajima variety} =$$

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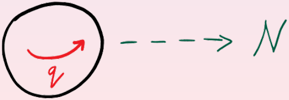
Input by Okounkov:

$$\text{q} - \text{difference equations for vertex functions} =$$
$$\text{qKZ equations} + \text{ dynamical equations}$$

through the study of quasimap moduli spaces for Nakajima varities



In the following we will talk about this in the simplest case:

- Nakajima variety: $N = T^*Gr(k, n)$
- Quantum Integrable System: $\mathfrak{sl}(2)$ XXZ spin chain.

Notation

$$T^*Gr(k,n) = N_{k,n}, \quad \sqcup_k N_{k,n} = N(n).$$

As a Nakajima variety:

$$N_{k,n} = T^*\mathcal{M}////GL(V) = \mu^{-1}(0)_s/GL(V),$$

where

$$T^*\mathcal{M} = Hom(V,W) \oplus Hom(W,V)$$

Tautological bundles:

$$\mathcal{V} = T^*\mathcal{M} \times V////GL(V), \quad \mathcal{W} = T^*\mathcal{M} \times W////GL(V)$$

For any $\tau \in K_{GL(V)}(\cdot) = \Lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_k^{\pm 1})$ we introduce a tautological bundle:

$$\tau = T^*\mathcal{M} \times \tau(V)////GL(V)$$

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Tori, Fixed points and Bethe roots

Torus action:

$$A = \mathbb{C}^\times_{a_1} \times \cdots \times \mathbb{C}^\times_{a_n} \circlearrowleft W,$$

$$\text{Full torus}: \ \ T = A \times \mathbb{C}^\times_\hbar, \ \text{where } \mathbb{C}^\times_\hbar \text{ scales cotangent directions}$$

Fixed points: $\mathbf{p} = \{s_1, \dots, s_k\} \in \{a_1, \dots, a_n\}$

Denote $\mathcal{A} := \mathbb{Q}(a_1, \dots, a_n, \hbar)$, $R := \mathbb{Z}(a_1, \dots, a_n, \hbar)$, then localized K-theory is:

$$K_T(N(n))_{loc} = K_T(N(n)) \otimes_R \mathcal{A} = \sum_{k=0}^{n} K_T(N_{k,n}) \otimes_R \mathcal{A}$$

is a 2^n-dimensional $\mathcal{A}$-vector space (Hilbert space for spin chain), spanned by $\mathcal{O}_\mathbf{p}$.

Classical Bethe equations: The eigenvalues of the operators of multiplication by τ are $\tau(x_1, \cdots, x_k)$ evaluated at the solutions of the following equations:

$$\prod_{j=1}^{n} (x_i - a_j) = 0, \quad i = 1, \dots, k, \ with \ x_i \neq x_j$$

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Quantum tautological classes and Bethe equations

We will use theory of quasimaps:

$$\mathcal{C} - - - \to N_{k,n}$$

in order to deform tensor product: $A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B \; z^d$.

We will also define quantum tautological classes:

$$\hat{\tau}(z) = \tau + \sum_{d=1}^{\infty} \tau_d z^d \in K_T(N(n))[[z]]$$

Theorem. [P. Pushkar, A. Smirnov, A.Z] The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1, \dots, x_k)$ evaluated at the solutions of the following equations:

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in order to deform tensor product: $A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B \; z^d$.

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Quantum tautological classes and Bethe equations

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The quantum K-theoretic meaning of the Q-operator

Theorem. [P. Pushkar, A. Smirnov, A.Z.]

- The quantum multiplication on quantum tautological class corresponding to $\tau_u := \oplus_{m\geq 0} u^m \Lambda^m \mathcal{V}$ coincides with Q-operator, i..e

$$\hat{\tau}_u(z) = Q(u)$$

- Explicit universal formulas for quantum products::

$$\widehat{\Lambda^\ell \mathcal{V}}(z) = \Lambda^\ell \mathcal{V} + a_1(z)\ F_0 \Lambda^{\ell-1} \mathcal{V} E_{-1} + \cdots + a_\ell(z)\ F_0^\ell E_{-1}^\ell,$$

where $a_m(z) = \frac{(\hbar-1)^m\ \hbar^{\frac{m(m+1)}{2}}\ K^m}{(m)_\hbar!\ \prod_{i=1}^{m}(1-(-1)^n z^{-1}\hbar^i K)}$,

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$\hbar$-oper connections for simple Lie groups

A $(G, \hbar)$-oper on $\mathbb{P}^1$ is a triple:

- $\mathcal{F}_G$ is a principal G-bundle
- $\mathcal{F}_B$ its reduction $\mathcal{F}_B$ to B
- $A \in Hom_{\mathcal{O}(\mathbb{P}^1)}(\mathcal{F}_G, \mathcal{F}_G^{(\hbar)})$ such that for any $C \in Hom_{\mathcal{O}(\mathbb{P}^1)}(\mathcal{F}_B, \mathcal{F}_B^{(\hbar)})$, the expression $C^{-1}A \in Hom_{\mathcal{O}(\mathbb{P}^1)}(\mathcal{F}_G, \mathcal{F}_G)$ takes values in $M^s = BsB$, $s = \prod_i s_i$ is a Coxeter element.

$$\text{Locally}: A(u) = n'(u)\prod_i(\phi_i^{\check{\alpha}_i}s_i)n(u), \ \ \phi_i \in \mathbb{C}, \ \ n(u), n'(u) \in N(u)$$

- $(G, \hbar)$-oper with *regular singularities* at finitely many points on $\mathbb{P}^1$:

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(based on work with P. Koroteev, D. Sage (2018)
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If G is of ADE type, then:

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In the non-simply-laced case we get different Bethe equations, not $\mathfrak{g}^L$!
Conjecturally corresponding to twisted affine Lie algebras.

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SL(2) example

A Z-twisted $(SL(2), \hbar)$-*oper* on $\mathbb{P}^1$ with regular singularities is a triple $(E, A, \mathcal{L})$:

- (E, A) is a $(SL(2), \hbar)$-connection
- $\mathcal{L}$ is a line subbundle so that $\bar{A} : \mathcal{L} \to (E/\mathcal{L})^q$ is an isomorphism except for zeroes of $\Lambda(u)$.
- A is gauge equivalent to $Z \in H$

Equivalently:

$$s(\hbar u) \wedge A(u) s(u) = \Lambda(u),$$

where $s(u)$ is a section of $\mathcal{L}$. Choosing trivialization $s(u) = \begin{pmatrix} Q_-(u) \\ Q_+(u) \end{pmatrix}$, we obtain that above condition is the QQ-system:

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Quantum q-Langlands correspondence

These two geometric descriptions are related , and illustrate the critical version of the quantum q-Langlands correspondence (outlined by M. Aganagic, E. Frenkel, A, Okounkov) :

One-to-one correspondence between:

- Conformal blocks for $\hbar$-deformed W-algebra, which are solutions to $\hbar$-difference equations emerging from $\hbar$-opers with regular singularities,
- Conformal blocks for $U_\hbar(\hat{\mathfrak{g}})$, solutions to qKZ equations.

Thank you!