Decorated super-Teichmueller spaces and super-Ptolemy relations

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AMS Sectional Metting: Special Session on Canonical Bases, Cluster Structures and Non-commutative Birational Geometry

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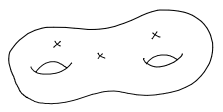
Outline

Introduction

Let $F_s^g \equiv F$ be the Riemann surface of genus g and s punctures.
We assume $s > 0$ and $2 - 2g - s < 0$.



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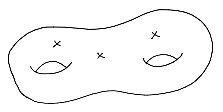

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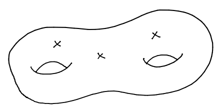

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Representation-theoretic definition:

$$T(F) \; = \; \mathrm{Hom}'(\pi_1(F), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}),$$

where $\rho \in Hom'$ if

- ρ is injective
- identity in $PSL(2,\mathbb{R})$ is not an accumulation point of the image of ρ, i.e. ρ is discrete
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The image $\Gamma \in PSL(2, \mathbb{R})$ is a *Fuchsian group*.

By Poincaré uniformization we have $F = H^+/\Gamma$, where $PSL(2, \mathbb{R})$ acts on the hyperbolic upper-half plane H^+ as oriented isometries, given by fractional-linear transformations:

$$z \to \frac{az+b}{cz+d}.$$

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The primary object of interest in many areas of mathematics is the *moduli space*:

$$M(F) = T(F)/MC(F).$$

The *mapping class group* $MC(F)$: a group of the homotopy classes of orientation preserving homeomorphisms.

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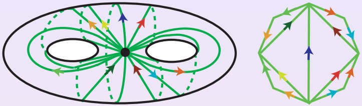
R. Penner's work in the 1980s: a construction of coordinates associated to the ideal triangulation of F:



so that one assigns one positive number λ-length for every edge.

This provides coordinates for the decorated Teichmüller space:

$$\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$$

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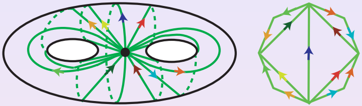

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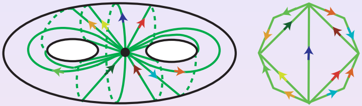

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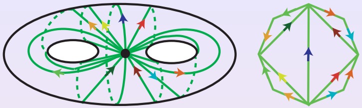

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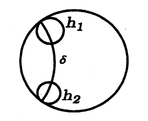

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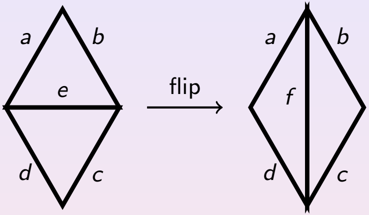

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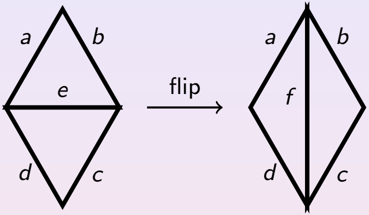

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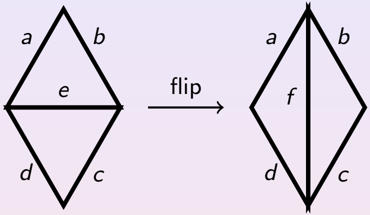

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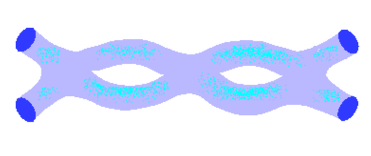

Superstrings, which, according to string theory, are the fundamental objects for the description of our world, carry extra anticommuting parameters θ^i, called *fermions*:

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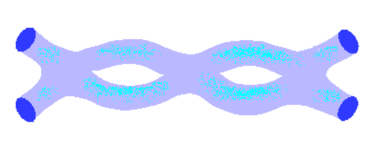

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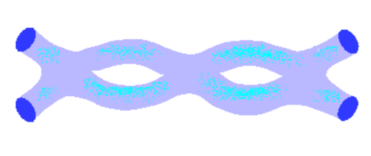

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The corresponding supermoduli spaces were intensively studied by various physicists and mathematicians L. Crane, J. Rabin, E. D'Hocker, D. Phong, A. Schwarz, A. Voronov...

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$$OSP(1|2), \quad OSP(2|2)$$

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Cast of Characters

i) Superspaces and supermanifolds

Let $\Lambda(\mathbb{K}) = \Lambda^0(\mathbb{K}) \oplus \Lambda^1(\mathbb{K})$ be an exterior algebra over field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with (in)finitely many generators 1, e_1, $e_2, \ldots$, so that

$$a = a^{\#} + \sum_i a_i e_i + \sum_{ij} a_{ij} e_i \wedge e_j + \ldots, \quad \# : \Lambda(\mathbb{K}) \to \mathbb{K}$$

$a^{\#}$ is referred to as a *body* of a supernumber.

If $a \in \Lambda^0(\mathbb{K})$, it is called even (bosonic) number

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Note, that odd numbers anticommute.

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Superspace $\mathbb{K}^{(n|m)}$ is:

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One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as *even and odd coordinates*.

Special spaces:

- Upper $\mathcal{N}=N$ super-half-plane (we will need $\mathcal{N}=1,2$):

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$$\mathbb{K}^{(n|m)} = \{(z_1, z_2, \ldots, z_n|\theta_1, \theta_2, \ldots, \theta_m) : z_i \in \Lambda^0(\mathbb{K}),\ \theta_j \in \Lambda^1(\mathbb{K})\}$$

One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as *even and odd coordinates*.

Special spaces:

- Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):

$$H^+ = \{(z|\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{C}^{(1|N)}| \ \mathrm{Im}\ z^\# > 0\}$$

- Positive superspace:

$$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \ldots, z_n|\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^{(n|m)}| \ z_i^\# > 0, i = 1, \ldots, n\}$$

ii) Supergroup $OSp(1|2)$

<u>Definition</u>:

$(1|2) \times (1|2)$ supermatrices g, obeying the relation

$$g^{st} J g = J,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the supertranspose g^{st} of g is given by

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{implies} \quad g^{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

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Some remarks:

• Lie superalgebra $osp(1|2)$:

Three even $h, X_\pm$ and two odd $v_\pm$ generators, satisfying the following commutation relations:

$$[h, v_\pm] = \pm v_\pm, \quad [v_\pm, v_\pm] = \mp 2X_\pm, \quad [v_+, v_-] = h.$$

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$OSp(1|2)$ acts on $\mathcal{N}=1$ super half-plane H^+, with the absolute $\partial H^+ = \mathbb{R}^{1|1}$ by superconformal fractional-linear transformations:

$$z \to \frac{az+b}{cz+d} + \eta \frac{\gamma z + \delta}{(cz+d)^2},$$
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Factor H^+/Γ, where Γ is a discrete subgroup of $OSp(1|2)$, such that its projection is a Fuchsian group, are called *super Riemann surfaces*.

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Alternatively, *super Riemann surface* is a complex $(1|1)$-supermanifold S with everywhere non-integrable odd distribution $\mathcal{D} \in TS$, such that

$$0 \to \mathcal{D} \to TS \to \mathcal{D}^2 \to 0 \quad \text{is} \quad \text{exact.}$$

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iii) ($\mathcal{N}=1$) Super-Teichmüller space

From now on let

$$ST(F) = \mathrm{Hom}'(\pi_1(F), OSp(1|2))/OSp(1|2).$$

Super-Fuchsian representations comprising Hom' are defined to be those whose projections

$$\pi_1 \to OSp(1|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

are Fuchsian groups, corresponding to F.

Trivial bundle $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$ is called the decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space, $ST(F)$ ($S\tilde{T}(F)$) has 2^{2g+s-1} connected components labeled by spin structures on F.

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iv) Ideal triangulations and trivalent fatgraphs

• Ideal triangulation of F: triangulation Δ of F with punctures at the vertices, so that each arc connecting punctures is not homotopic to a point rel punctures.

• Trivalent fatgraph: trivalent graph τ with cyclic orderings on half-edges about each vertex.

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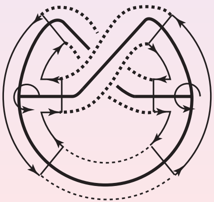
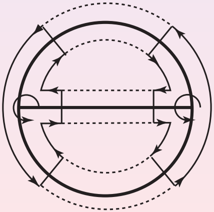


Fatgraph for F_1^1



Fatgraph for F_0^3

v) Spin structures

Textbook definition:

Let M be an oriented n-dimensional Riemannian manifold, P_{SO} is an orthonormal frame bundle, associated with TM. A *spin structure* is a 2-fold covering map $P \to P_{SO}$, which restricts to $Spin(n) \to SO(n)$ on each fiber.

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This is not really useful for us, since we want to relate it to combinatorial geometric structures on F.

There are several ways to describe spin structures on F:

- D. Johnson (1980):

Quadratic forms $q : H_1(F, \mathbb{Z}_2) \to \mathbb{Z}_2$, which are quadratic with respect to the intersection pairing $\cdot : H_1 \otimes H_1 \to \mathbb{Z}_2$, i.e. $q(a + b) = q(a) + q(b) + a \cdot b$ if $a, b \in H_1$.

- S. Natanzon:

A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \to SL(2, \mathbb{R})$ of $\rho : \pi_1 \to PSL_2(\mathbb{R})$. Quadratic form q is computed using the following rules: $\mathrm{trace}\, \tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.

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- D. Cimasoni and N. Reshetikhin (2007):

Combinatorial description of spin structures in terms of the so-called Kasteleyn orientations and dimer configurations on the one-skeleton of a suitable CW decomposition of F. They derive a formula for the quadratic form in terms of that combinatorial data.

• We gave a substantial simplification of the combinatorial formulation of spin structures on F (one of the main results of R. Penner, A. Zeitlin, arXiv:1509.06302):

Equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips:

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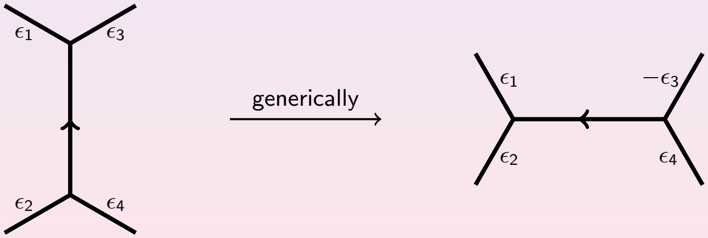

Coordinates on $S\tilde{T}(F)$

Fix a surface $F = F_g^s$ as above and

- $\tau \subset F$ is some trivalent fatgraph spine
- ω is an orientation on the edges of τ whose class in $\mathcal{O}(\tau)$ determines the component C of $S\tilde{T}(F)$

Then there are global affine coordinates on C:

- one even coordinate called a λ-length for each edge
- one odd coordinate called a μ-invariant for each vertex of τ,

the latter of which are taken modulo an overall change of sign.

Alternating the sign in one of the fermions corresponds to the reflection on the spin graph.

The above λ-lengths and μ-invariants establish a real-analytic homeomorphism

$$C \to \mathbb{R}_+^{6g-6+3s|4g-4+2s}/\mathbb{Z}_2.$$

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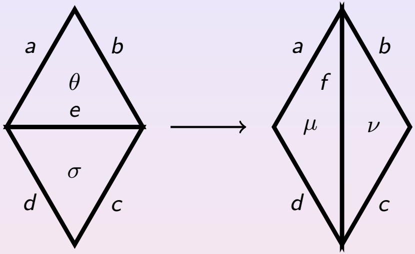
Superflips

When all a, b, c, d are different edges of the triangulations of F,



Ptolemy transformations are as follows:

$$ef = (ac + bd)\Big(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big),$$

$$\nu = \frac{\sigma + \theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1+\chi}}.$$

$\chi = \frac{ac}{bd}$ denotes the cross-ratio, and the evolution of spin graph follows from the construction associated to the spin graph evolution rule.

• These coordinates are natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{\Gamma} \in S\tilde{T}(F)$, then $\tilde{\varphi}(\tilde{\Gamma})$ is determined by the orientation and coordinates on edges and vertices of $\varphi(\tau)$ induced by φ from the orientation ω, the λ-lengths and μ-invariants on τ.

• There is an even 2-form on $S\tilde{T}(F)$ which is invariant under super Ptolemy transformations, namely,

$$\omega = \sum_v d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2,$$

where the sum is over all vertices v of τ where the consecutive half edges incident on v in clockwise order have induced λ-lengths a, b, c and θ is the μ-invariant of v.

• Coordinates on $ST(F)$:

Take instead of λ-lengths shear coordinates $z_e = \log\left(\frac{ac}{bd}\right)$ for every edge e, which are subject to linear relation: the sum of all z_e adjacent to a given vertex $= 0$.

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Sketch of construction via hyperbolic supergeometry

XIXth century perspective on hyperbolic (super)geometry:

$OSp(1|2)$ acts on super-Minkowski space $\mathbb{R}^{2,1|2}$ (in the bosonic case $PSL(2,\mathbb{R})$ acts on $\mathbb{R}^{2,1}$).

If $A = (x_1, x_2, y, \phi, \theta)$ and $A' = (x_1', x_2', y', \phi', \theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A' \rangle = \frac{1}{2}(x_1 x_2' + x_1' x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- Superhyperboloid $\mathbb{H}$ consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$
- Positive super light cone L^+ consisting of points $B \in \mathbb{R}^{2,1|2}$ satisfying $\langle B, B \rangle = 0$,

where $x_1^{\#}, x_2^{\#} \geq 0$.

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$OSp(1|2)$ does not act transitively on L^+:

The space of orbits is labelled by odd variable up to a sign.

We pick an orbit of the vector $(1, 0, 0, 0, 0)$ and denote it L_0^+.

There is an equivariant projection from L_0^+ to $\mathbb{R}^{1|1} = \partial H^+$.

Goal: Construction of the π_1-equivariant lift for all the data from the universal cover $\tilde{F}$, associated to its triangulation to L_0^+.

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Orbits of 2 and 3 points in L_0^+

- There is a unique $OSp(1|2)$-invariant of two linearly independent vectors $A, B \in L_0^+$, and it is given by the pairing $\langle A, B\rangle$, the square root of which we will call λ-length.

Let $\zeta^b\zeta^e\zeta^a$ be a positive triple in L_0^+. Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even r, s, t, which have positive bodies, and odd θ so that

$$g\cdot\zeta^e = t(1,1,1,\theta,\theta),\ \ g\cdot\zeta^b = r(0,1,0,0,0),\ \ g\cdot\zeta^a = s(1,0,0,0,0).$$

- The moduli space of $OSp(1|2)$-orbits of positive triples in the light cone is given by $(a, b, e, \theta) \in \mathbb{R}_+^{3|1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by fermionic reflection.

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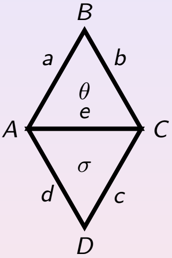
Orbits of 4 points in L_0^+: basic calculation

Suppose points A, B, C are put in the standard position.

The 4th point D, so that two new λ- lengths are c, d.



Fixing the sign of θ, we fix the sign of Manin invariant σ in terms of coordinates of D.

Important observation: if we turn the picture upside down, then

$$(\theta, \sigma) \to (\sigma, -\theta)$$

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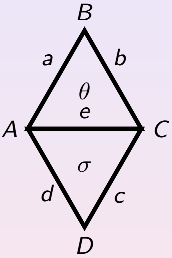

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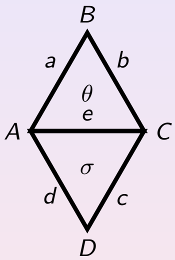

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The lift of ideal triangulation to super-Minkowski space

Denote:

- Δ is ideal trangulation of F, $\tilde{\Delta}$ is ideal triangulation of the universal cover $\tilde{F}$
- Δ_∞ ($\tilde{\Delta}_\infty$)-collection of ideal points of F ($\tilde{F}$).

Consider Δ together with:

• the orientation on the fatgraph $\tau(\Delta)$,

coordinate system $\tilde{C}(F, \Delta)$, i.e.

• positive even coordinate for every edge

• odd coordinate for every triangle

We call coordinate vectors $\vec{c}$, $\vec{c'}$ equivalent if they are identical up to overall reflection of sign of odd coordinates.

Let $C(F, \Delta) \equiv \tilde{C}(F, \Delta)/\sim$. This implies that

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for every quadrilateral $ABCD$, if the arrow is pointing from σ to θ then the lift is given by the picture from the previous slide up to post-composition with the element of $OSp(1|2)$.

The construction of ℓ can be done in a recursive way:

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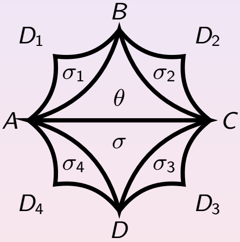

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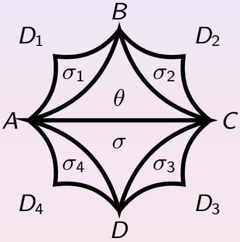

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Theorem

Fix $F, \Delta, \tau(\Delta)$ as before. Let ω be an orientation, corresponding to a specified spin structure s of F. Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

$$\ell_\omega : \tilde{\Delta}_\infty \to L_0^+$$

which is uniquely determined up to post-composition by $OSp(1|2)$ under admissibility conditions discussed above, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates.

There is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \to OSp(1|2)$, uniquely determined up to conjugacy by an element of $OSp(1|2)$ such that

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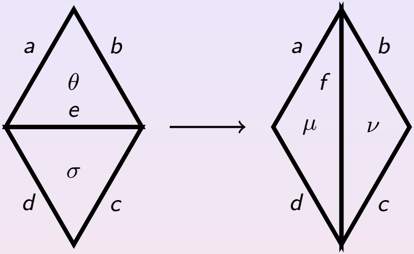
The super-Ptolemy transformations



$$ef = (ac+bd)\Big(1+\frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big),$$
$$\nu = \frac{\sigma+\theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi}-\theta}{\sqrt{1+\chi}}$$

are the consequence of light cone geometry.

The space of all such lifts ℓ_ω coincides with the decorated super-Teichmüller space $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$.

In order to remove the decoration, one can pass to shear coordinates $z_e = \log\left(\frac{ac}{bd}\right)$.

It is easy to check that the 2-form

$$\omega = \sum_{\Delta} d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2$$

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Ramond Decoration

Further reduction of the decoration: $S\tilde{T}(F)=\mathbb{R}_+^{6g+3s-6|4g+2s-4}/\mathbb{Z}_2$ is actually an $\mathbb{R}_+^{(s|n_R)}$-decoration over physically relevant Teichmüller space.

Here n_R is the number of Ramond punctures, which means that the small contour γ surrounding the puncture is such that $q[\gamma]=1$, i.e. $tr(\tilde{\rho}(\gamma)>0$.

On the level of hyperbolic geometry, the appropriate constraint is that the monodromy group element has to be true parabolic, i.e. to be conjugated to the parabolic element of $SL(2,\mathbb{R})$ subgroup.

We formulated it in terms of invariant constraints on shear coordinates in:

I. Ip, R. Penner, A. Z., arXiv:1709.06207, Comm. Math. Phys. 371 (2019) 145-157, arXiv:1709.06207

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Here n_R is the number of Ramond punctures, which means that the small contour γ surrounding the puncture is such that $q[\gamma] = 1$, i.e. $tr(\tilde{\rho}(\gamma) > 0$.

On the level of hyperbolic geometry, the appropriate constraint is that the monodromy group element has to be true parabolic, i.e. to be conjugated to the parabolic element of $SL(2, \mathbb{R})$ subgroup.

We formulated it in terms of invariant constraints on shear coordinates in:

I. Ip, R. Penner, A. Z., arXiv:1709.06207, Comm. Math. Phys. 371 (2019) 145-157, arXiv:1709.06207

$\mathcal{N}=2$ super-Teichmüller theory: prerequisites

$\mathcal{N}=2$ super-Teichmüller space is related to $OSP(2|2)$ supergroup of rank 2.

It is more useful to work with its 3×3 incarnation, which is isomorphic to $\Psi \ltimes SL(1|2)_0$, where Ψ is a certain automorphism of the Lie algebra $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$.

$SL(1|2)_0$ is a supergroup, consisting of supermatrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix}$$

such that $f > 0$ and their Berezinian $= 1$.

This group acts on the space $\mathbb{C}^{1|2}$ as superconformal franctional-linear transformations.

As before, $\mathcal{N}=2$ super-Fuchsian groups are the ones whose projections

$$\pi_1 \to OSP(2|2) \to GL^+(2,\mathbb{R}) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

are Fuchsian.

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Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2,\mathbb{R})$.

Therefore, the construction of coordinates requires a new notion: $\mathbb{R}_+$-graph connection.

A G-graph connection on τ is the assignment $h_e \in G$ to each oriented edge e of τ so that $h_{\bar{e}} = h_e^{-1}$ if $\bar{e}$ is the opposite orientation to e. Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex v of τ such that $h'_e = t_v h_e t_w^{-1}$ for each oriented edge $e \in \tau$ with initial point v and terminal point w.

The moduli space of flat G-connections on F is isomorphic to the space of equivalent G-graph connections on τ.

By the way, spin structures can be identified with equivalence classes of $\mathbb{Z}_2$-graph connections.

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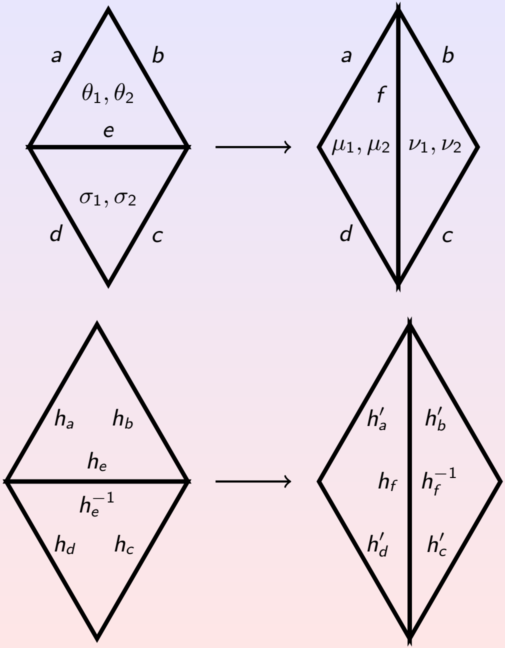
Data on triangulation/fatgraphs:

- One positive parameter per edge of fatgraph/triangulation
- Two odd parameters per triangle
- Two spin structures: generated by reflection of signs and the permutation of odd parameters
- $\mathbb{R}_+$-graph connection

Generic Ptolemy transformations are:



and the transformation formulas are as follows:

$$ef = (ac+bd)\left(1 + \frac{h_e^{-1}\sigma_1\theta_2}{2(\sqrt{\chi}+\sqrt{\chi}^{-1})} + \frac{h_e\sigma_2\theta_1}{2(\sqrt{\chi}+\sqrt{\chi}^{-1})}\right),$$

$$\mu_1 = \frac{h_e\theta_1 + \sqrt{\chi}\sigma_1}{\mathcal{D}}, \quad \mu_2 = \frac{h_e^{-1}\theta_2 + \sqrt{\chi}\sigma_2}{\mathcal{D}},$$

$$\nu_1 = \frac{\sigma_1 - \sqrt{\chi}h_e\theta_1}{\mathcal{D}}, \quad \nu_2 = \frac{\sigma_2 - \sqrt{\chi}h_e^{-1}\theta_2}{\mathcal{D}},$$

$$h_a' = \frac{h_a}{h_e c_\theta}, \quad h_b' = \frac{h_b c_\theta}{h_e}, \quad h_c' = h_c\frac{c_\theta}{c_\mu}, \quad h_d' = h_d\frac{c_\nu}{c_\theta}, \quad h_f = \frac{c_\sigma}{c_\theta^2},$$

where

$$\mathcal{D} := \sqrt{1 + \chi + \frac{\sqrt{\chi}}{2}(h_e^{-1}\sigma_1\theta_2 + h_e\sigma_2\theta_1)},$$

$$c_\theta := 1 + \frac{\theta_1\theta_2}{6}.$$

There is a parallel construction, based on Jenkins-Strebel differentials.

How to glue a Riemann surface based on a fatgraph with the metric data?

Jenkins-Strebel differential and the underlying fatgraph→

special covering of Riemann surfaces with double overlaps, corresponding to the edges.

M. Kontsevich'92; M. Mulase, M. Penkava'98

In a joint work with A. Schwarz, we

- Explicitly construct deformations for the class of (1|1)-supermanifolds "of middle degree" with punctures as Čech cocycles
- Get in contact with the analogue of Penner's convex hull construction
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McShane-type identities, path to volumes?

The simplest McShane identity (G. McShane'92):

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on a cusped torus, where sum is over all simple geodesics γ and ℓ_γ is the length.

M. Mirzakhani used such types of identities to deal with the volumes of the moduli spaces.

Y. Huang recently shown how to deal with McShane identities using Penner's lambda length coordinates.

Together with Y. Huang, R. Penner, we have shown that the following generalization of McShane identity holds:

$$\frac{1}{2}=\sum_{\gamma}\Big(\frac{1}{1+e^{\ell_\gamma}}+\frac{W_\gamma}{4}\frac{\sinh(\frac{\ell_\gamma}{2})}{\cosh^2(\frac{\ell_\gamma}{2})}\Big)$$

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Open problems/directions

1) Cluster superalgebras

2) Weil-Petersson-form in $\mathcal{N} = 2$ case

3) Quantization of super-Teichmüller spaces

4) Analogues of Weil-Petersson volumes

5) Relation to Strebel theory

6) Quasi-abelianization to $GL(1|1)$/spectral network approach in the style of Gaiotto-Moore-Neitzke

Thank you!