Hyperbolic supergeometry, Super-Teichmueller spaces and applications

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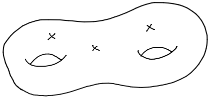
Outline

Let $F_s^g \equiv F$ be the Riemann surface of genus g and s punctures. We assume $s > 0$ and $2 - 2g - s < 0$.



$$T(F) \;=\; \mathrm{Hom}'(\pi_1(F), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}),$$

where $\rho \in Hom'$ if

- ρ is injective
- identity in $PSL(2,\mathbb{R})$ is not an accumulation point of the image of ρ, i.e. ρ is discrete
- the group elements corresponding to loops around punctures are parabolic ($|\mathrm{tr}| = 2$)

Replacing $PSL(2, R)$ with

$$OSP(1|2), \quad OSP(2|2)$$

one obtains $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Teichmüller spaces.

In the late 80s the problem of construction of Penner's coordinates on $ST(F)$ was introduced on Yu.I. Manin's Moscow seminar.

The $\mathcal{N} = 1$ case was solved in:
R. Penner, A. Z., J. Diff. Geom. 111 (2019) 527-566, arXiv:1509.06302

The $\mathcal{N} = 2$ case was solved in:
I. Ip, R. Penner, A. Z., Adv. Math. 336 (2018) 409-454, arXiv:1605.08094

Full decoration removal for $\mathcal{N} = 1$:
I. Ip, R. Penner, A. Z., arXiv:1709.06207, Comm. Math. Phys. 371 (2019) 145-157, arXiv:1709.06207

Super McShane idenity:
Y. Huang, R. Penner, A. Z., arXiv:1907.09978

Hyperbolic Supergeometry

Coordinates on Super-Teichmüller space

Super McShane identity

N=2 super-Teichmüller theory

Open problems

Supernumbers

Let $\Lambda(\mathbb{K}) = \Lambda^0(\mathbb{K}) \oplus \Lambda^1(\mathbb{K})$ be an exterior algebra over field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with (in)finitely many generators 1, e_1, $e_2, \ldots$, so that

$$a = a^{\#} + \sum_i a_i e_i + \sum_{ij} a_{ij} e_i \wedge e_j + \ldots, \quad \# : \Lambda(\mathbb{K}) \to \mathbb{K}$$

$a^{\#}$ is referred to as a *body* of a supernumber.

If $a \in \Lambda^0(\mathbb{K})$, it is called even (bosonic) number

If $a \in \Lambda^1(\mathbb{K})$, it is called odd (fermionic) number

Note, that odd numbers anticommute.

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Superspaces and supermanifolds

Superspace $\mathbb{K}^{(n|m)}$ is:

$$\mathbb{K}^{(n|m)} = \{(z_1, z_2, \ldots, z_n|\theta_1, \theta_2, \ldots, \theta_m) : z_i \in \Lambda^0(\mathbb{K}),\ \theta_j \in \Lambda^1(\mathbb{K})\}$$

One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as *even and odd coordinates*.

Special spaces:

- Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):

$$H^+ = \{(z|\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{C}^{(1|N)} |\ \mathrm{Im}\ z^\# > 0\}$$

- Positive superspace:

$$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \ldots, z_n|\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^{(n|m)} |\ z_i^\# > 0, i = 1, \ldots, n\}$$

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Supergroup $OSp(1|2)$

A subgroup of $GL(1|2)$, namely invertible $(1|2)\times(1|2)$ supermatrices g, obeying the relation:

$$g^{st}Jg = J,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the supertranspose g^{st} of g is given by

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{implies} \quad g^{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

We want a connected component of identity, so we assume that Berezinian (super-analogue of determinant) $= 1$.

Important remark: Note, that the *body* of the supergroup $OSP(1|2)$ is $SL(2,\mathbb{R})$, not $PSL(2,\mathbb{R})$!

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Lie superalgebra $osp(1|2)$

Three even

$$h,\ X_\pm$$

and two odd

$$v_\pm$$

generators, satisfying the following commutation relations:

$$[h, v_\pm] = \pm v_\pm, \quad [v_\pm, v_\pm] = \mp 2X_\pm, \quad [v_+, v_-] = h.$$

An important observation is that Killing form gives a super-Minkowski space $\mathbb{R}^{2,1|2}$.

Hyperbolic supergeometry

$OSp(1|2)$ acts on super-Minkowski space $\mathbb{R}^{2,1|2}$ (in the bosonic case $PSL(2,\mathbb{R})$ acts on $\mathbb{R}^{2,1}$).

If $A = (x_1, x_2, y|\phi, \theta)$ and $A' = (x_1', x_2', y'|\phi', \theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A' \rangle = \frac{1}{2}(x_1 x_2' + x_1' x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- Superhyperboloid $\mathbb{H}$ consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$
- Positive super light cone L^+ consisting of points $B \in \mathbb{R}^{2,1|2}$ satisfying $\langle B, B \rangle = 0$,

where $x_1^\#, x_2^\# \geq 0$.

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$OSp(1|2)$-action on the upper half-plane and super-Riemann surfaces

$OSp(1|2)$ acts on $\mathcal{N}=1$ super half-plane H^+, with the absolute $\partial H^+ = \mathbb{R}^{1|1}$ by superconformal fractional-linear transformations:

$$z \to \frac{az+b}{cz+d} + \eta\frac{\gamma z+\delta}{(cz+d)^2},$$
$$\eta \to \frac{\gamma z+\delta}{cz+d} + \eta\frac{1+\frac{1}{2}\delta\gamma}{cz+d}.$$

Factor H^+/Γ, where Γ is a discrete subgroup of $OSp(1|2)$, such that its projection is a Fuchsian group, are called *super Riemann surfaces*.

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Alternate characterization of super-Riemann surfaces

Alternatively, *super Riemann surface* is a complex $(1|1)$-supermanifold S with everywhere non-integrable odd distribution $\mathcal{D} \in TS$, such that

$$0 \to \mathcal{D} \to TS \to \mathcal{D}^2 \to 0 \quad \text{is} \quad \text{exact.}$$

There are more general fractional-linear transformations acting on H^+. They correspond to $SL(1|2)$ supergroup, and factors H^+/Γ give $(1|1)$-supermanifolds which have relation to $\mathcal{N} = 2$ super-Teichmüller theory.

The light cone

$OSp(1|2)$ does not act transitively on L^+:

The space of orbits is labelled by odd variable up to a sign: $L^+ = \cup_{|\theta|} L^+_{|\theta|}$.

We pick an orbit of the vector $(1, 0, 0|0, \theta)$ and denote it $L^+_{|\theta|}$.

There is an equivariant projection from L^+_0 to $\mathbb{R}^{1|1} = \partial H^+$.

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Geodesics

- Special geodesics: the ones with endpoints on the rays of L_0^+ (or on $\mathbb{R}^{1|1}$). They become pure bosonic under $OSp(1|2)$ action.
- General geodesics: endpoints are labeled by fermions up to a sign: $|\alpha, \beta|$.

Explicit expression:

$$\mathbf{x}(t) = \mathbf{u}\ ch(t) + \mathbf{v}\ sh(t),$$

where $<\mathbf{u}, \mathbf{u}>= 1$, $<\mathbf{v}, \mathbf{v}>= -1$, $<\mathbf{u}, \mathbf{v}>= 0$.

Here t is a length parameter, $\mathbf{e} = \mathbf{u} + \mathbf{v}$, $\mathbf{f} = \mathbf{u} - \mathbf{v}$ generate the light cone rays at the endpoints which belong to the orbits $L^+_{|\alpha|}$, $L^+_{|\beta|}$.

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Triangles in $L^+_{|\theta|}$ and their invariants.

• There is a unique $OSp(1|2)$-invariant of two linearly independent vectors $A, B \in L^+_0$, and it is given by the pairing $\langle A, B\rangle$, the square root of which we will call λ-length.

• The moduli space of $OSp(1|2)$-orbits of positive triples in the light cone is given by $(a, b, e|\alpha, \beta, \epsilon, \theta) \in \mathbb{R}^{3|4}_+/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by fermionic reflection.

Let $\zeta^b\zeta^e\zeta^a$ be a positive triple in L^+_0, then $\alpha, \beta, \epsilon = 0$. Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even r, s, t, which have positive bodies, and odd θ so that

$$g \cdot \zeta^e = t(1, 1, 1|\theta, \theta), \; g \cdot \zeta^b = r(0, 1, 0|0, 0), \; g \cdot \zeta^a = s(1, 0, 0|0, 0).$$

On the superline $\mathbb{R}^{1|1}$ the parameter θ is known as *Manin invariant*.

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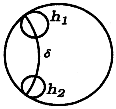
Horocycles and λ-lengths

Key notion for Penner coordinates: horocycles.

These are (1|2)-dimensional spaces determined by $\mathbf{u} \in L_0^+$:

$$h(\mathbf{u}) = \{\mathbf{P} \in \mathbb{H} : \ \langle \mathbf{P}, \mathbf{u} \rangle = \frac{1}{\sqrt{2}}\}$$

Positive parameters correspond to the "renormalized" geodesic lengths:



The lambda length $\lambda = e^{\delta/2} = \sqrt{\langle h(\mathbf{u}_1), h(\mathbf{u}_2) \rangle}$

Mapping class group action

Moduli space:

$$M(F) = T(F)/MC(F).$$

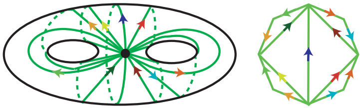
The *mapping class group* $MC(F)$: a group of the homotopy classes of orientation preserving homeomorphisms.

$MC(F)$ acts on $T(F)$ by outer automorphisms of $\pi_1(F)$.

The goal is to find a system of coordinates on $T(F)$, so that the action of $MC(F)$ is realized in the simplest possible way.

Penner coordinates in the standard situation

R. Penner's work in the 1980s: a construction of coordinates associated to the ideal triangulation of F:



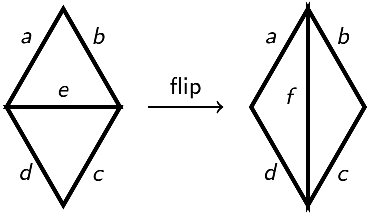
so that one assigns one positive number λ-length for every edge.

This construction provides coordinates for the decorated Teichmüller space:

$$\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$$

Ptolemy transformations

The action of $MC(F)$ can be described combinatorially using elementary transformations called flips:



$$\text{Ptolemy relation}: \qquad ef = ac + bd$$

In order to obtain coordinates on $T(F)$, one has to consider *shear coordinates* $z_e = \log(\frac{ac}{bd})$, which are subjects to certain linear constraints.

$\mathcal{N}=1$ Super-Teichmüller space

From now on let

$$ST(F) = \mathrm{Hom}'(\pi_1(F), OSp(1|2))/OSp(1|2).$$

Super-Fuchsian representations comprising Hom' are defined to be those whose projections

$$\pi_1 \to OSp(1|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

are Fuchsian groups, corresponding to F.

Trivial bundle $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$ is called the decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space, $ST(F)$ ($S\tilde{T}(F)$) has 2^{2g+s-1} connected components labeled by spin structures on F.

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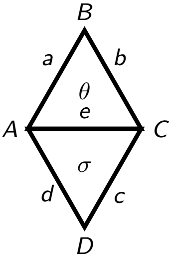
Orbits of 4 points in L_0^+: basic calculation

Suppose points A, B, C are put in the standard position.

The 4th point D, so that two new λ- lengths are c, d.



Fixing the sign of θ, we fix the sign of Manin invariant σ in terms of coordinates of D.

Important observation: if we turn the picture upside down, then

$$(\theta, \sigma) \to (\sigma, -\theta)$$

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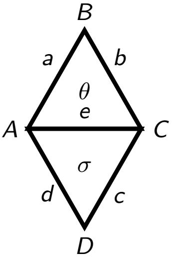

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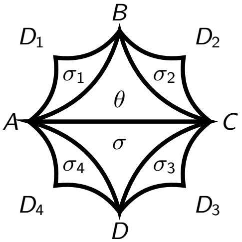
Recursive procedure and equivariant lift

For every quadrilateral $ABCD$, if there is a direction from σ to θ then the lift is given by the picture from the previous slide up to post-composition with the element of $OSp(1|2)$.

The construction of lift ℓ from H^+ with data to Minkowski space can be done in a recursive way:



Such lift is unique up to post-composition with $OSp(1|2)$ group element and it is π_1-equivariant. This allows us to construct representation of π_1 in $OSP(1|2)$, based on the provided data.

Ideal triangulations and trivalent fatgraphs

Dual to each other:

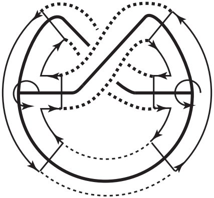
• Ideal triangulation of F: triangulation Δ of F with punctures at the vertices, so that each arc connecting punctures is not homotopic to a point rel punctures.

• Trivalent fatgraph: trivalent graph τ with cyclic orderings on half-edges about each vertex.

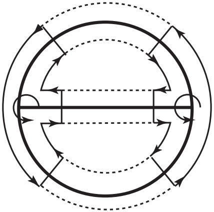
$\tau = \tau(\Delta)$, if the folowing is true:

1) one fatgraph vertex per triangle

2) one edge of fatgraph intersects one shared edge of triangulation.



Fatgraph for F^1



Fatgraph for F^3

Spin structures

There are several ways to describe spin structures on F:

- D. Johnson (1980):

  Quadratic forms $q : H_1(F, \mathbb{Z}_2) \to \mathbb{Z}_2$, which are quadratic with respect to the intersection pairing $\cdot : H_1 \otimes H_1 \to \mathbb{Z}_2$, i.e. $q(a + b) = q(a) + q(b) + a \cdot b$ if $a, b \in H_1$.

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  A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \to SL(2, \mathbb{R})$ of $\rho : \pi_1 \to PSL_2(\mathbb{R})$. Quadratic form q is computed using the following rules: $\mathrm{trace}\ \tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.

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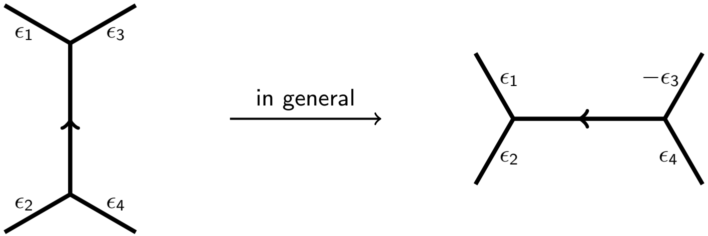
Simplest combinatorial description

We gave a substantial simplification of the combinatorial formulation of spin structures on F (one of the main results of R. Penner, A. Zeitlin, arXiv:1509.06302):

Equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips:



Coordinates on $S\tilde{T}(F)$

Fix a surface $F = F_g^s$ as above and

- $\tau \subset F$ is some trivalent fatgraph spine
- ω is an orientation on the edges of τ whose class in $\mathcal{O}(\tau)$ determines the component C of $S\tilde{T}(F)$

Then there are global affine coordinates on C:

- one even coordinate called a λ-length for each edge
- one odd coordinate called a μ-invariant for each vertex of τ,

the latter of which are taken modulo an overall change of sign.

Alternating the sign in one of the fermions corresponds to the reflection on the spin graph.

The above λ-lengths and μ-invariants establish a real-analytic homeomorphism

$$C \to \mathbb{R}_+^{6g-6+3s|4g-4+2s}/\mathbb{Z}_2.$$

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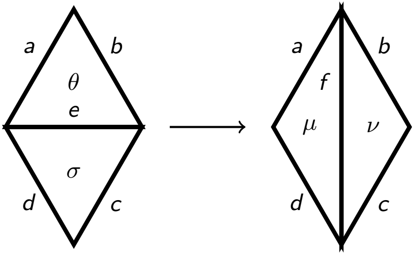
Superflips

When all a, b, c, d are different edges of the triangulations of F,



Ptolemy transformations are as follows:

$$ef = (ac + bd)\left(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\right),$$

$$\nu = \frac{\sigma + \theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1+\chi}}.$$

$\chi = \frac{ac}{bd}$ denotes the cross-ratio, and the evolution of spin graph follows from the construction associated to the spin graph evolution rule.

• These coordinates are natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{\Gamma} \in S\tilde{T}(F)$, then $\tilde{\varphi}(\tilde{\Gamma})$ is determined by the orientation and coordinates on edges and vertices of $\varphi(\tau)$ induced by φ from the orientation ω, the λ-lengths and μ-invariants on τ.

• There is an even 2-form on $S\tilde{T}(F)$ which is invariant under super Ptolemy transformations, namely,

$$\omega = \sum_v d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2,$$

where the sum is over all vertices v of τ where the consecutive half edges incident on v in clockwise order have induced λ-lengths a, b, c and θ is the μ-invariant of v.

• Coordinates on $ST(F)$:

Take instead of λ-lengths shear coordinates $z_e = \log\left(\frac{ac}{bd}\right)$ for every edge e, which are subject to linear relation: the sum of all z_e adjacent to a given vertex $= 0$.

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Ramond Decoration

Further reduction of the decoration: $S\tilde{T}(F) = \mathbb{R}_+^{6g+3s-6|4g+2s-4}/\mathbb{Z}_2$ is actually an $\mathbb{R}_+^{(s|n_R)}$-decoration over physically relevant Teichmüller space.

Here n_R is the number of Ramond punctures, which means that the small contour γ surrounding the puncture is such that $q[\gamma] = 1$, i.e. $tr(\tilde{\rho}(\gamma) > 0$.

On the level of hyperbolic geometry, the appropriate constraint is that the monodromy group element has to be true parabolic, i.e. to be conjugated to the parabolic element of $SL(2, \mathbb{R})$ subgroup.

We formulated it in terms of invariant constraints on shear coordinates in:

I. Ip, R. Penner, A. Z., arXiv:1709.06207, Comm. Math. Phys. 371 (2019) 145-157, arXiv:1709.06207

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McShane identity

The McShane identity for 1-puctured torus (G. McShane'92):

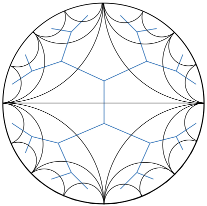
$$\frac{1}{2} = \sum_{\gamma} \frac{1}{1 + e^{\ell_\gamma}}$$

on a cusped torus, where the sum is over all simple geodesics γ and ℓ_γ is the length.

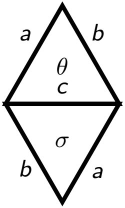
There are many ways to prove it. One proof was given by B.H. Bowditch'96, which uses the so-called Markov triples:

$$a^2 + b^2 + c^2 = abc,$$

the fact that $T(F)$ is identified with the Poincare disk, and the cell complex dual to Farey tessalation:



Super analogue of Bowditch construction



Denote $W_c = \theta\sigma$ if the arrow is oriented from σ to θ.

$$a^2 + b^2 + c^2 + abW_c + aW_b + bcW_a = habc,$$

where h is an invariant we call $super\ semi-perimeter$.

Ptolemy relation/edge relation: $cd = a^2 + b^2 + abW_c$

The length of the geodesic could be read from the group element:

$$|str(g_a) + 1| = 2cosh(\ell_{\gamma_a}/2) = r_a + {r_a}^{-1} = ah - W_a$$

Super McShane identity

The identity:

$$\sum_a \left(\frac{1}{ahr_a} + \frac{W_a}{2ah}\right) = \frac{1}{2},$$

which translates into:

$$\sum_\gamma \left[\frac{1}{1+e^{\ell_\gamma}} + \frac{W_\gamma}{4}\frac{\sinh(\frac{\ell_\gamma}{2})}{\cosh^2(\frac{\ell_\gamma}{2})}\right] = \frac{1}{2}$$

where ℓ_γ is the superanalogue of geodesic length and W_γ is a product of μ-coordinates.

Y. Huang, R. Penner, A. Z., arXiv:1907.09978

Fatgraphs and super-Riemann surfaces

There is a parallel construction, based on Jenkins-Strebel differentials.

How to glue a Riemann surface based on a fatgraph with the metric data?

Jenkins-Strebel differential and the underlying fatgraph$\rightarrow$

special covering of Riemann surfaces with double overlaps, corresponding to the edges.

M. Kontsevich'92; M. Mulase, M. Penkava'98

In a joint work with A. Schwarz, we

- Explicitly construct deformations for the class of (1|1)-supermanifolds "of middle degree" with punctures as Čech cocycles
- Get in contact with the analogue of Penner's convex hull construction
- Construct N=1 SRS using the dualities of (1|1)-supermanifolds/$N = 2$ SRS

$\mathcal{N} = 2$ super-Teichmüller theory: prerequisites

$\mathcal{N} = 2$ super-Teichmüller space is related to $OSP(2|2)$ supergroup of rank 2.

It is more useful to work with its 3×3 incarnation, which is isomorphic to $\Psi \ltimes SL(1|2)_0$, where Ψ is a certain automorphism of the Lie algebra $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$.

$SL(1|2)_0$ is a supergroup, consisting of supermatrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix}$$

such that $f > 0$ and their Berezinian $= 1$.

This group acts on the space $\mathbb{C}^{1|2}$ as superconformal franctional-linear transformations.

As before, $\mathcal{N} = 2$ super-Fuchsian groups are the ones whose projections

$$\pi_1 \to OSP(2|2) \to GL^+(2, \mathbb{R}) \to SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$$

are Fuchsian.

Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2,\mathbb{R})$.

Therefore, the construction of coordinates requires a new notion: $\mathbb{R}_+$-graph connection.

A G-graph connection on τ is the assignment $h_e \in G$ to each oriented edge e of τ so that $h_{\bar{e}} = h_e^{-1}$ if $\bar{e}$ is the opposite orientation to e. Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex v of τ such that $h'_e = t_v h_e t_w^{-1}$ for each oriented edge $e \in \tau$ with initial point v and terminal point w.

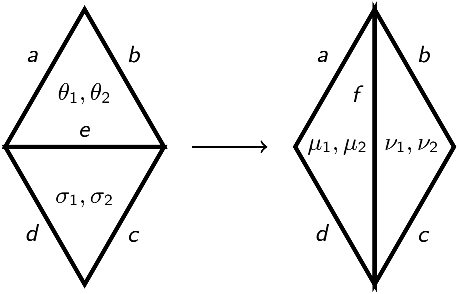
The moduli space of flat G-connections on F is isomorphic to the space of equivalent G-graph connections on τ.

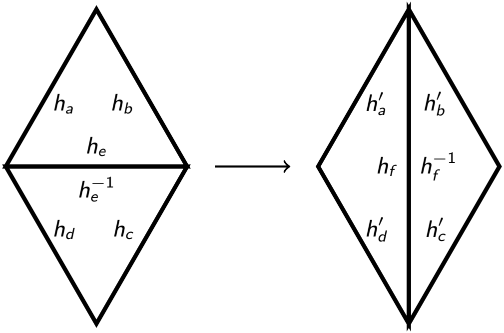
By the way, spin structures can be identified with equivalence classes of $\mathbb{Z}_2$-graph connections.

Data on triangulation/fatgraphs:

- One positive parameter per edge of fatgraph/triangulation
- Two odd parameters per triangle
- Two spin structures: generated by reflection of signs and the permutation of odd parameters
- $\mathbb{R}_+$-graph connection

Generic Ptolemy transformations are:





and the transformation formulas are as follows:

$$ef = (ac+bd)\left(1+\frac{h_e^{-1}\sigma_1\theta_2}{2(\sqrt{\chi}+\sqrt{\chi}^{-1})}+\frac{h_e\sigma_2\theta_1}{2(\sqrt{\chi}+\sqrt{\chi}^{-1})}\right),$$

$$\mu_1=\frac{h_e\theta_1+\sqrt{\chi}\sigma_1}{\mathcal{D}},\quad \mu_2=\frac{h_e^{-1}\theta_2+\sqrt{\chi}\sigma_2}{\mathcal{D}},$$

$$\nu_1=\frac{\sigma_1-\sqrt{\chi}h_e\theta_1}{\mathcal{D}},\quad \nu_2=\frac{\sigma_2-\sqrt{\chi}h_e^{-1}\theta_2}{\mathcal{D}},$$

$$h'_a=\frac{h_a}{h_e c_\theta},\quad h'_b=\frac{h_b c_\theta}{h_e},\quad h'_c=h_c\frac{c_\theta}{c_\mu},\quad h'_d=h_d\frac{c_\nu}{c_\theta},\quad h_f=\frac{c_\sigma}{c_\theta^2},$$

where

$$\mathcal{D}:=\sqrt{1+\chi+\frac{\sqrt{\chi}}{2}(h_e^{-1}\sigma_1\theta_2+h_e\sigma_2\theta_1)},$$

$$c_\theta:=1+\frac{\theta_1\theta_2}{6}.$$

Open problems/directions

1) Cluster superalgebras

2) Weil-Petersson-form in $\mathcal{N} = 2$ case

3) Quantization of super-Teichmüller spaces

4) Analogues of Weil-Petersson volumes

5) Relation to Strebel theory

6) Quasi-abelianization to $GL(1|1)$/spectral network approach in the style of Gaiotto-Moore-Neitzke

Thank you!