Quantum/classical duality and geometry

Anton M. Zeitlin

Louisiana State University

University of South Alabama

Mobile

October 14, 2023

Example of integrable system: harmonic oscillator

2-dimensional phase space with coordinate q and momentum p:

Hamiltonian: $$H = \frac{p^2 + q^2}{2},$$

Poisson bracket: $$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}.$$

Equations of motion: $$\left.\begin{aligned} \tfrac{dq}{dt} &= \{H, q\} = p \\ \tfrac{dp}{dt} &= \{H, p\} = -q \end{aligned}\right\} \Rightarrow \frac{d^2 q}{dt^2} + q = 0$$

Action-angle variables: polar coordinates in (q, p)-space.

Energy level set: $L_E = \{p^2 + q^2 = E\}$ is a circle.

Equations of motion for action-angle variables (H, ϕ):

$$\frac{d\phi}{dt} = \omega, \quad \frac{dH}{dt} = 0$$

Example of integrable system: harmonic oscillator

2-dimensional phase space with coordinate q and momentum p:

Hamiltonian: $$H = \frac{p^2 + q^2}{2},$$

Poisson bracket: $$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}.$$

Equations of motion: $$\left.\begin{array}{l} \frac{dq}{dt} = \{H, q\} = p \\ \frac{dp}{dt} = \{H, p\} = -q \end{array}\right\} \Rightarrow \frac{d^2 q}{dt^2} + q = 0$$

Action-angle variables: polar coordinates in (q, p)-space.

Energy level set: $L_E = \{p^2 + q^2 = E\}$ is a circle.

Equations of motion for action-angle variables (H, ϕ):

$$\frac{d\phi}{dt} = \omega, \quad \frac{dH}{dt} = 0$$

Example of integrable system: harmonic oscillator

2-dimensional phase space with coordinate q and momentum p:

$$\text{Hamiltonian:} \qquad H = \frac{p^2 + q^2}{2},$$

$$\text{Poisson bracket:} \qquad \{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}.$$

$$\text{Equations of motion:} \qquad \left.\begin{array}{l} \frac{dq}{dt} = \{H, q\} = p \\ \frac{dp}{dt} = \{H, p\} = -q \end{array}\right\} \Rightarrow \frac{d^2 q}{dt^2} + q = 0$$

Action-angle variables: polar coordinates in (q, p)-space.

Energy level set: $L_E = \{p^2 + q^2 = E\}$ is a circle.

Equations of motion for action-angle variables (H, ϕ):

$$\frac{d\phi}{dt} = \omega, \qquad \frac{dH}{dt} = 0$$

Example of integrable system: harmonic oscillator

2-dimensional phase space with coordinate q and momentum p:

Hamiltonian: $$H = \frac{p^2+q^2}{2},$$

Poisson bracket: $$\{F,G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}.$$

Equations of motion: $$\left.\begin{aligned}\tfrac{dq}{dt} &= \{H,q\} = p\\ \tfrac{dp}{dt} &= \{H,p\} = -q\end{aligned}\right\} \Rightarrow \frac{d^2q}{dt^2} + q = 0$$

Action-angle variables: polar coordinates in (q,p)-space.

Energy level set: $L_E = \{p^2+q^2 = E\}$ is a circle.

Equations of motion for action-angle variables (H,ϕ):

$$\frac{d\phi}{dt} = \omega, \quad \frac{dH}{dt} = 0$$

Classical integrable systems: what are they?

Symplectic $2n$-manifold M: phase space, which has information of coordinates and momenta of a physical system.

Equations of motion:

$$\frac{df}{dt} = \{H, f\}.$$

Integrability: family of conserved quantities: $\{F_i\}_{i=1}^n$:

$$\{F_i, F_j\} = 0, \quad F_1 = H.$$

Liouville-Arnold theorem:

- Compact connected components of $L_c = \{F_i = c_i\}_{i=1}^n$ are diffeomorphic to $\mathbb{T}^n$.
- Existence of action-angle variables $\{I_i\}_{i=1}^n$, $\{\phi^i\}_{i=1}^n$ in the neighborhood of L_c:

$$\frac{d\phi^i}{dt} = \omega^i, \quad \frac{dI_i}{dt} = 0.$$

Finding action/angle variables is a non-trivial problem.

Classical integrable systems: what are they?

Symplectic $2n$-manifold M: phase space, which has information of coordinates and momenta of a physical system.

Equations of motion:

$$\frac{df}{dt} = \{H, f\}.$$

Integrability: family of conserved quantities: $\{F_i\}_{i=1}^n$:

$$\{F_i, F_j\} = 0, \quad F_1 = H.$$

Liouville-Arnold theorem:

- Compact connected components of $L_c = \{F_i = c_i\}_{i=1}^n$ are diffeomorphic to $\mathbb{T}^n$.
- Existence of action-angle variables $\{I_i\}_{i=1}^n$, $\{\phi^i\}_{i=1}^n$ in the neighborhood of L_c:

$$\frac{d\phi^i}{dt} = \omega^i, \quad \frac{dI_i}{dt} = 0.$$

Finding action/angle variables is a non-trivial problem.

Explosion of interest in integrable systems: 60s -70s

Integrable soliton equations in (1+1)-dimensions,
e.g. Korteweg-de Vries (KdV) equation:

$$u_t = -u_{xxx} + 6uu_x.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura'67; P. Lax'68;

L. Faddeev, V. Zakharov'71

Lie-theoretic methods through Lax pair formulation:

$$\frac{dL}{dt} = [A, L],$$

where $L = -\partial_x^2 + u(x,t)$ for KdV.

I. Gelfand, L. Dickey'76; V. Drinfeld, V. Sokolov'85

Inverse Scattering Method (ISM):

$$\text{spectral data of } L \to \text{action-angle variables}$$

At the same time many finite-dimensional multiparticle integrable systems were discovered: Calogero-Moser, Toda, Ruijsennars-Schneider, etc.

Explosion of interest in integrable systems: 60s -70s

Integrable soliton equations in (1+1)-dimensions,
e.g. Korteweg-de Vries (KdV) equation:

$$u_t = -u_{xxx} + 6uu_x.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura'67; P. Lax'68;

L. Faddeev, V. Zakharov'71

Lie-theoretic methods through Lax pair formulation:

$$\frac{dL}{dt} = [A, L],$$

where $L = -\partial_x^2 + u(x,t)$ for KdV.

I. Gelfand, L. Dickey'76; V. Drinfeld, V. Sokolov'85

Inverse Scattering Method (ISM):

spectral data of $L \to$ action-angle variables

At the same time many finite-dimensional multiparticle integrable systems were discovered: Calogero-Moser, Toda, Ruijsennars-Schneider, etc.

Explosion of interest in integrable systems: 60s -70s

Integrable soliton equations in (1+1)-dimensions,
e.g. Korteweg-de Vries (KdV) equation:

$$u_t = -u_{xxx} + 6uu_x.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura'67; P. Lax'68;

L. Faddeev, V. Zakharov'71

Lie-theoretic methods through Lax pair formulation:

$$\frac{dL}{dt} = [A, L],$$

where $L = -\partial_x^2 + u(x,t)$ for KdV.

I. Gelfand, L. Dickey'76; V. Drinfeld, V. Sokolov'85

Inverse Scattering Method (ISM):

spectral data of $L \to$ action-angle variables

At the same time many finite-dimensional multiparticle integrable systems were discovered: Calogero-Moser, Toda, Ruijsennars-Schneider, etc.

Explosion of interest in integrable systems: 60s -70s

Integrable soliton equations in (1+1)-dimensions,
e.g. Korteweg-de Vries (KdV) equation:

$$u_t = -u_{xxx} + 6uu_x.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura'67; P. Lax'68;

L. Faddeev, V. Zakharov'71

Lie-theoretic methods through Lax pair formulation:

$$\frac{dL}{dt} = [A, L],$$

where $L = -\partial_x^2 + u(x, t)$ for KdV.

I. Gelfand, L. Dickey'76; V. Drinfeld, V. Sokolov'85

Inverse Scattering Method (ISM):

spectral data of $L \rightarrow$ action-angle variables

At the same time many finite-dimensional multiparticle integrable systems were discovered: Calogero-Moser, Toda, Ruijsennars-Schneider, etc.

Quantum Integrable models: 70s-80s

Quantum integrability:

$$[H_i, H_j] = 0, \quad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables $\to$ simultaneous diagonalization of H_i.

Quantization of (1+1)-models? Put them on the lattice.

Lattice integrable models $\to$ new algebraic structures:

R-matrix and Yang-Baxter equation

accompanied with

algebraic Bethe ansatz

lead to the the discovery of Quantum inverse scattering method (QISM) developed by Leningrad School.

That eventually led to the discovery of quantum groups by Drinfeld and Jimbo.

Quantum Integrable models: 70s-80s

Quantum integrability:

$$[H_i, H_j] = 0, \quad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables $\to$ simultaneous diagonalization of H_i.

Quantization of (1+1)-models? Put them on the lattice.

Lattice integrable models $\to$ new algebraic structures:

R-matrix and Yang-Baxter equation

accompanied with

algebraic Bethe ansatz

lead to the the discovery of Quantum inverse scattering method (QISM) developed by Leningrad School.

That eventually led to the discovery of quantum groups by Drinfeld and Jimbo.

Quantum integrability:

$$[H_i, H_j] = 0, \quad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables $\to$ simultaneous diagonalization of H_i.

Quantization of (1+1)-models? Put them on the lattice.

Lattice integrable models $\to$ new algebraic structures:

R-matrix and Yang-Baxter equation

accompanied with

algebraic Bethe ansatz

lead to the the discovery of Quantum inverse scattering method (QISM) developed by Leningrad School.

That eventually led to the discovery of quantum groups by Drinfeld and Jimbo.

Quantum Integrable models: 70s-80s

Quantum integrability:

$$[H_i, H_j] = 0, \quad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables $\to$ simultaneous diagonalization of H_i.

Quantization of (1+1)-models? Put them on the lattice.

Lattice integrable models $\to$ new algebraic structures:

R-matrix and Yang-Baxter equation

accompanied with

algebraic Bethe ansatz

lead to the the discovery of Quantum inverse scattering method (QISM) developed by Leningrad School.

That eventually led to the discovery of quantum groups by Drinfeld and Jimbo.

90s: Geometrization era begins

- Dubrovin, Givental, Kontsevich, Witten established first relations with integrability in the context of enumerative geometry.

  Notable cases:

  - Witten's conjecture, proven by Kontsevich, relating intersection numbers on the moduli space of curves and the τ-function of KdV model.
  - Givental and collaborators: description of the enumerative geometry of flag varieties (quantum cohomology/quantum K-theory) via classical and quantum multiparticle systems of Toda type.

- Feigin, Frenkel, and collaborators, while studying conformal field theory/representation theory of affine Lie algebras, discovered the relation:

  Connections on $\mathbb{P}^1$ called opers $\leftrightarrow$ Gaudin integrable model

  That turned out to be an example of the geometric Langlands correspondence.

90s: Geometrization era begins

- Dubrovin, Givental, Kontsevich, Witten established first relations with integrability in the context of enumerative geometry.

  Notable cases:

  - Witten's conjecture, proven by Kontsevich, relating intersection numbers on the moduli space of curves and the τ-function of KdV model.
  - Givental and collaborators: description of the enumerative geometry of flag varieties (quantum cohomology/quantum K-theory) via classical and quantum multiparticle systems of Toda type.

- Feigin, Frenkel, and collaborators, while studying conformal field theory/representation theory of affine Lie algebras, discovered the relation:

  Connections on $\mathbb{P}^1$ called opers $\leftrightarrow$ Gaudin integrable model

  That turned out to be an example of the geometric Langlands correspondence.

2000s – geometric representation theory and integrable models

- Nakajima, Schiffmann, Varagnolo, Vasserot...:

  Geometric realization of representations of quantum groups on cohomology and K-theory of symplectic resolutions, in particular, on Nakajima quiver varieties.

  Okounkov:
  "Symplectic resolutions are the Lie algebras of XXI century"

- 2010s: Nekrasov, Shatashvili:

  Hints from supersymmetric gauge theory $\rightarrow$ geometric realization of quantum integrable models solved by Bethe ansatz.

  Okounkov and his school: enumerative geometry of symplectic resolutions.

- Nakajima, Schiffmann, Varagnolo, Vasserot...:

  Geometric realization of representations of quantum groups on cohomology and K-theory of symplectic resolutions, in particular, on Nakajima quiver varieties.

  Okounkov:
  "Symplectic resolutions are the Lie algebras of XXI century"

- 2010s: Nekrasov, Shatashvili:

  Hints from supersymmetric gauge theory $\to$ geometric realization of quantum integrable models solved by Bethe ansatz.

  Okounkov and his school: enumerative geometry of symplectic resolutions.

In this talk

- Duality between:
  - Finite-dimensional classical integrable systems: (trigonometric) Ruijsennaars-Schenider, Calogero-Moser;
  - Quantum integrable models based on quantum groups: XXZ, XXX, (trigonometric) Gaudin quantum integrable models.
- Geometric interpretation and applications.

In collaboration with:

P. Koroteev, P. Pushkar, E. Frenkel, D. Sage, A. Smirnov:

- P. Koroteev, A.Z., *The Zoo of Opers and Dualities*,

  arXiv:2208.08031
- P. Koroteev, A. Z., *3d Mirror Symmetry for Instanton Moduli Spaces*,

  Comm. Math. Phys.'23
- P. Koroteev, A.Z., *q-Opers, QQ-Systems, and Bethe Ansatz II: Generalized Minors*,

  J. Reine Angew. Math.'23
- P. Koroteev, A.Z.,*Toroidal q-Opers*,

  J. Inst. Math. Jussieu'23
- P. Koroteev, E. Frenkel, D. Sage, A.Z., *q-Opers, QQ-Systems, and Bethe Ansatz*,

  J. Eur. Math. Soc., arXiv:2002.07344
- P. Koroteev, D. Sage, A.Z., *(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality*,

  Comm. Math. Phys.'21
- P. Koroteev, P. Pushkar, A. Smirnov, A.Z., *Quantum K-theory of Quiver Varieties and Many-Body Systems*,

  Selecta Math. '21
- P. Koroteev, A.Z., *qKZ/tRS Duality via Quantum K-Theoretic Counts*,

  Math. Res. Lett.'21
- P. Pushkar, A. Smirnov, A.Z.,*Baxter Q-operator from quantum K-theory*,

  Adv. Math.'20

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules*, form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
- a_i are values for t

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules*, form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
- a_i are values for t

Quantum groups:

$$U_q(\hat{\mathfrak{g}})$$

are deformations of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1) \otimes V_2(a_2)$$

$$V_2(a_2) \otimes V_1(a_1)$$

Quantum groups

Quantum groups:

$$U_q(\hat{\mathfrak{g}})$$

are deformations of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1) \otimes V_2(a_2)$$

$$V_2(a_2) \otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:

The generators of $U_q(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices: FRT construction.

Spin chain models and transfer matrices

Physical space:

$$\mathcal{H}_{\text{phys}} = V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n).$$

Auxiliary spaces: $\{W(u)\}$.

Quantum monodromy matrix:

$$M(u) = (Z \otimes \mathrm{Id})\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\text{phys}} \to W(u) \otimes \mathcal{H}_{\text{phys}}$$

Here $\tilde{R}$ is the R-matrix, composed with permutation operator,
$Z \in e^{\mathfrak{h}}$ - diagonal.

Physical space:

$$\mathcal{H}_{\rm phys} = V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n).$$

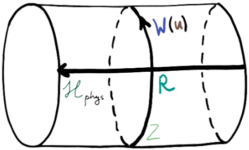
Auxiliary spaces: $W(u)$.

Quantum monodromy matrix:

$$M(u) = (Z \otimes \mathrm{Id})\ \tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

Here $\tilde{R}$ is the R-matrix, composed with permutation operator, $Z \in e^{\mathfrak{h}} \in U_q(\hat{\mathfrak{g}})$ - diagonal.

Transfer matrix:



$$T_{W(u)} = \mathrm{Tr}_{W(u)}\Big[M(u)\Big], \quad T_{W(u)} : \mathcal{H}_{\rm phys} \to \mathcal{H}_{\rm phys}$$

Integrability:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

follows from Yang-Baxter relation.

Transfer matrices $T_{W(u)}$ generate Bethe algebra:

$$T_{W(u)} = \sum_n u^n I_n, \qquad [I_n, I_m] = 0.$$

Primary goal: diagonalize $\{T_{W(u)}\}$ simultaneously.

Tool: Bethe ansatz.

Integrability:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

follows from Yang-Baxter relation.

Transfer matrices $T_{W(u)}$ generate Bethe algebra:

$$T_{W(u)} = \sum_n u^n I_n, \qquad [I_n, I_m] = 0.$$

Primary goal: diagonalize $\{T_{W(u)}\}$ simultaneously.

Tool: Bethe ansatz.

Various points of view on Bethe ansatz

- via Algebraic Bethe ansatz:

  Central for the QISM.

  Developed in Leningrad: late 70s-80s

- via Frenkel-Reshetikhin (qKZ) equation:

  I. Frenkel, N. Reshetikhin '92

  Geometrization through enumerative geometry of quiver varieties:

  N. Nekrasov, S. Shatashvili '09, A. Okounkov '15; A. Okounkov, A. Smirnov '16; M. Aganagic, A. Okounkov '17;

  P. Pushkar, A. Smirnov, A.Z. '16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z .'17

- via QQ-systems:

  appeared first in the context of qKdV equation and ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov'98; D. Masoero, A. Raimondo, D. Valeri'16; Frenkel, Hernandez '13,'19

  Geometric interpretation of QQ-systems through the difference analogue of connections on the projective line, the so-called (G, q)-opers.

  joint work with E. Frenkel, P. Koroteev, D. Sage '18 – '22

Various points of view on Bethe ansatz

- via Algebraic Bethe ansatz:

  Central for the QISM.

  Developed in Leningrad: late 70s-80s

- via Frenkel-Reshetikhin (qKZ) equation:

  I. Frenkel, N. Reshetikhin '92

  Geometrization through enumerative geometry of quiver varieties:

  N. Nekrasov, S. Shatashvili '09, A. Okounkov '15; A. Okounkov, A. Smirnov '16; M. Aganagic, A. Okounkov '17;

  P. Pushkar, A. Smirnov, A.Z. '16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z .'17

- via QQ-systems:

  appeared first in the context of qKdV equation and ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov'98; D. Masoero, A. Raimondo, D. Valeri'16; Frenkel, Hernandez '13,'19

  Geometric interpretation of QQ-systems through the difference analogue of connections on the projective line, the so-called (G, q)-opers.

  joint work with E. Frenkel, P. Koroteev, D. Sage '18 – '22

Various points of view on Bethe ansatz

- via Algebraic Bethe ansatz:

  Central for the QISM.

  Developed in Leningrad: late 70s-80s

- via Frenkel-Reshetikhin (qKZ) equation:

  I. Frenkel, N. Reshetikhin '92

  Geometrization through enumerative geometry of quiver varieties:
  N. Nekrasov, S. Shatashvili '09, A. Okounkov '15; A. Okounkov, A. Smirnov '16; M. Aganagic, A. Okounkov '17;

  P. Pushkar, A. Smirnov, A.Z. '16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z .'17

- via QQ-systems:

  appeared first in the context of qKdV equation and ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov'98; D. Masoero, A. Raimondo, D. Valeri'16; Frenkel, Hernandez '13,'19

  Geometric interpretation of QQ-systems through the difference analogue of connections on the projective line, the so-called (G,q)-opers.

  joint work with E. Frenkel, P. Koroteev, D. Sage '18 – '22

QQ-systems

Consider Lie algebra $\mathfrak{g}$ of rank r.

Cartan matrix: $\{a_{ij}\}_{i,j=1,\dots,r}$, $a_{ij} = \langle \check{\alpha}_i, \alpha_j \rangle$.

QQ-system:

$$\widetilde{\xi}_i Q^i_-(u) Q^i_+(qu) - \xi_i Q^i_-(qu) Q^i_+(u) = \Lambda_i(u) \prod_{j \neq i} \left[\prod_{k=1}^{-a_{ij}} Q^j_+(q^{b^k_{ij}} u) \right]$$

$$i = 1, \dots, r, \quad b^k_{ij} \in \mathbb{Z}$$

$\{\Lambda_i(u), Q^i_\pm(u)\}_{i=1,\dots,r}$– polynomials, ξ_i, $\widetilde{\xi}_i$, $q \in \mathbb{C}^\times$;
$\{\Lambda_i(z)\}_{i=1,\dots,r}$-fixed.

Solving for $\{Q^i_+(z)\}_{i=1,\dots,r}$; $\{Q^i_-(z)\}_{i=1,\dots,r}$-auxiliary.

$$\text{If } \mathfrak{g} \text{ is of ADE type}: \begin{cases} b_{ij} = 1, & i > j \\ b_{ij} = 0, & i < j \end{cases}$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$:

$$\widetilde{\xi} Q_-(u) Q_+(qu) - \xi Q_-(qu) Q_+(u) = \Lambda(u).$$

In what context do they appear?

- Relations in the extended Grothendieck ring for finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; E. Frenkel, D. Hernandez '13,'19

- Bethe ansatz equations for XXX, XXZ models: $Q^i_\pm$ are eigenvalues of Baxter operators.

  E. Mukhin, A. Varchenko, ...

- Relations in quantum equivariant K-theory/quantum cohomology of quiver varieties. Baxter operators are generating functions of tautological bundles $\widehat{Q}^i_+(u) = \sum_{m=0}^n u^m \Lambda^m \mathcal{V}_i$.

  P. Pushkar, A. Smirnov, A.Z.'16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

- Spectral determinant relations in ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; D. Masoero, A. Raimondo, D. Valeri '16

- q-connections on the projective line: (G, q)-opers

  P. Koroteev, D. Sage, E. Frenkel, A.Z. '18; P. Koroteev, D. Sage, E. Frenkel, A.Z. '20;

  P. Koroteev, A.Z. '21; T. Brinson, D. Sage, A.Z. '21

In what context do they appear?

- Relations in the extended Grothendieck ring for finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; E. Frenkel, D. Hernandez '13,'19

- Bethe ansatz equations for XXX, XXZ models: $Q^i_{\pm}$ are eigenvalues of Baxter operators.

  E. Mukhin, A. Varchenko, . . .

- Relations in quantum equivariant K-theory/quantum cohomology of quiver varieties. Baxter operators are generating functions of tautological bundles $\widehat{Q}^i_+(u) = \sum_{m=0}^n u^m \Lambda^m \mathcal{V}_i$.

  P. Pushkar, A. Smirnov, A.Z.'16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

- Spectral determinant relations in ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; D. Masoero, A. Raimondo, D. Valeri '16

- q-connections on the projective line: (G, q)-opers

  P. Koroteev, D. Sage, E. Frenkel, A.Z. '18; P. Koroteev, D. Sage, E. Frenkel, A.Z. '20;

  P. Koroteev, A.Z. '21; T. Brinson, D. Sage, A.Z. '21

In what context do they appear?

- Relations in the extended Grothendieck ring for finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; E. Frenkel, D. Hernandez '13,'19

- Bethe ansatz equations for XXX, XXZ models: $Q^i_\pm$ are eigenvalues of Baxter operators.

  E. Mukhin, A. Varchenko, ...

- Relations in quantum equivariant K-theory/quantum cohomology of quiver varieties. Baxter operators are generating functions of tautological bundles $\widehat{Q}^i_+(u) = \sum_{m=0}^n u^m \Lambda^m \mathcal{V}_i$.

  P. Pushkar, A. Smirnov, A.Z.'16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

- Spectral determinant relations in ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; D. Masoero, A. Raimondo, D. Valeri '16

- q-connections on the projective line: (G, q)-opers

  P. Koroteev, D. Sage, E. Frenkel, A.Z. '18; P. Koroteev, D. Sage, E. Frenkel, A.Z. '20;

  P. Koroteev, A.Z. '21; T. Brinson, D. Sage, A.Z. '21

In what context do they appear?

- Relations in the extended Grothendieck ring for finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; E. Frenkel, D. Hernandez '13,'19

- Bethe ansatz equations for XXX, XXZ models: $Q^i_\pm$ are eigenvalues of Baxter operators.

  E. Mukhin, A. Varchenko, ...

- Relations in quantum equivariant K-theory/quantum cohomology of quiver varieties. Baxter operators are generating functions of tautological bundles $\widehat{Q}^i_+(u) = \sum_{m=0}^n u^m \Lambda^m \mathcal{V}_i$.

  P. Pushkar, A. Smirnov, A.Z.'16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

- Spectral determinant relations in ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; D. Masoero, A. Raimondo, D. Valeri '16

- q-connections on the projective line: (G, q)-opers

  P. Koroteev, D. Sage, E. Frenkel, A.Z. '18; P. Koroteev, D. Sage, E. Frenkel, A.Z. '20;

  P. Koroteev, A.Z. '21; T. Brinson, D. Sage, A.Z. '21

In what context do they appear?

- Relations in the extended Grothendieck ring for finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; E. Frenkel, D. Hernandez '13,'19

- Bethe ansatz equations for XXX, XXZ models: $Q^i_{\pm}$ are eigenvalues of Baxter operators.

  E. Mukhin, A. Varchenko, ...

- Relations in quantum equivariant K-theory/quantum cohomology of quiver varieties. Baxter operators are generating functions of tautological bundles $\widehat{Q}^i_+(u) = \sum_{m=0}^n u^m \Lambda^m \mathcal{V}_i$.

  P. Pushkar, A. Smirnov, A.Z.'16; P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

- Spectral determinant relations in ODE/IM correspondence

  V. Bazhanov, S. Lukyanov, A. Zamolodchikov '98; D. Masoero, A. Raimondo, D. Valeri '16

- q-connections on the projective line: (G, q)-opers

  P. Koroteev, D. Sage, E. Frenkel, A.Z. '18; P. Koroteev, D. Sage, E. Frenkel, A.Z. '20;

  P. Koroteev, A.Z. '21; T. Brinson, D. Sage, A.Z. '21

Miura (G, q)-opers and QQ-system

q-gauge transformation on $\mathbb{P}^1$:

$$A(u) \mapsto g(qu)A(u)g(u)^{-1},$$

where $A(u) \in G(u) \equiv G(\mathbb{C}(u))$.

Locally, with respect to reduction $\mathcal{F}_{B_-}$:

$$A(u) = n'(u)\prod_i \left[\phi_i(u)^{\check{\alpha}_i} s_i\right] n(u),\ \ \phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N_-(u)$$

Miura condition: another reduction $\mathcal{F}_{B_+}$ preserved.

(G, q)-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(qu)Zv^{-1}(u),\ \text{where}\ Z = \prod_i \zeta_i^{\check{\alpha}_i},\ \ v(u) \in G(u).$$

Regular singularities: $\phi_i(u) \equiv \Lambda_i(u) = \prod_i (u - a_i) \in \mathbb{C}[u])$.

Turns out that Z-twisted Miura (G, q)-oper with regular singularities has the form:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = \zeta_i \frac{Q^i_+(qu)}{Q^i_+(u)}.$$

Miura (G, q)-opers and QQ-system

q-gauge transformation on $\mathbb{P}^1$:

$$A(u) \mapsto g(qu)A(u)g(u)^{-1},$$

where $A(u) \in G(u) \equiv G(\mathbb{C}(u))$.

Locally, with respect to reduction $\mathcal{F}_{B_-}$:

$$A(u) = n'(u)\prod_i \left[\phi_i(u)^{\check{\alpha}_i} s_i\right] n(u),\ \ \phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N_-(u)$$

Miura condition: another reduction $\mathcal{F}_{B_+}$ preserved.

(G, q)-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(qu)Zv^{-1}(u),\ \text{where } Z = \prod_i \zeta_i^{\check{\alpha}_i},\ \ v(u) \in G(u).$$

Regular singularities: $\phi_i(u) \equiv \Lambda_i(u) = \prod_i (u - a_i) \in \mathbb{C}[u])$.

Turns out that Z-twisted Miura (G, q)-oper with regular singularities has the form:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = \zeta_i \frac{Q_+^i(qu)}{Q_+^i(u)}.$$

Miura (G, q)-opers and QQ-system

q-gauge transformation on $\mathbb{P}^1$:

$$A(u) \mapsto g(qu)A(u)g(u)^{-1},$$

where $A(u) \in G(u) \equiv G(\mathbb{C}(u))$.

Locally, with respect to reduction $\mathcal{F}_{B_-}$:

$$A(u) = n'(u) \prod_i \left[\phi_i(u)^{\check{\alpha}_i} s_i\right] n(u), \;\; \phi_i(u) \in \mathbb{C}(u), \;\; n(u), n'(u) \in N_-(u)$$

Miura condition: another reduction $\mathcal{F}_{B_+}$ preserved.

(G, q)-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(qu)Zv^{-1}(u), \;\; \text{where } Z = \prod_i \zeta_i^{\check{\alpha}_i}, \;\; v(u) \in G(u).$$

Regular singularities: $\phi_i(u) \equiv \Lambda_i(u) = \prod_i (u - a_i) \in \mathbb{C}[u]$).

Turns out that Z-twisted Miura (G, q)-oper with regular singularities has the form:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = \zeta_i \frac{Q_+^i(qu)}{Q_+^i(u)}.$$

Miura (G, q)-opers and QQ-system

q-gauge transformation on $\mathbb{P}^1$:

$$A(u) \mapsto g(qu)A(u)g(u)^{-1},$$

where $A(u) \in G(u) \equiv G(\mathbb{C}(u))$.

Locally, with respect to reduction $\mathcal{F}_{B_-}$:

$$A(u) = n'(u) \prod_i \left[\phi_i(u)^{\check{\alpha}_i} s_i\right] n(u), \;\; \phi_i(u) \in \mathbb{C}(u), \;\; n(u), n'(u) \in N_-(u)$$

Miura condition: another reduction $\mathcal{F}_{B_+}$ preserved.

(G, q)-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(qu)Zv^{-1}(u), \text{ where } Z = \prod_i \zeta_i^{\check{\alpha}_i}, \;\; v(u) \in G(u).$$

Regular singularities: $\phi_i(u) \equiv \Lambda_i(u) = \prod_i (u - a_i) \in \mathbb{C}[u])$.

Turns out that Z-twisted Miura (G, q)-oper with regular singularities has the form:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = \zeta_i \frac{Q_+^i(qu)}{Q_+^i(u)}.$$

Miura (G,q)-opers and QQ-system

q-gauge transformation on $\mathbb{P}^1$:

$$A(u) \mapsto g(qu)A(u)g(u)^{-1},$$

where $A(u) \in G(u) \equiv G(\mathbb{C}(u))$.

Locally, with respect to reduction $\mathcal{F}_{B\_}$:

$$A(u) = n'(u)\prod_i \left[\phi_i(u)^{\check{\alpha}_i} s_i\right] n(u),\ \ \phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N_-(u)$$

Miura condition: another reduction $\mathcal{F}_{B_+}$ preserved.

(G,q)-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(qu)Zv^{-1}(u),\ \ \text{where } Z = \prod_i \zeta_i^{\check{\alpha}_i},\ \ v(u) \in G(u).$$

Regular singularities: $\phi_i(u) \equiv \Lambda_i(u) = \prod_i (u - a_i) \in \mathbb{C}[u])$.

Turns out that Z-twisted Miura (G,q)-oper with regular singularities has the form:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = \zeta_i \frac{Q_+^i(qu)}{Q_+^i(u)}.$$

$(SL(r+1),q)$-opers: alternative definition

A $(GL(r+1),q)$-*oper* on $\mathbb{P}^1$ is a triple $(A,E,\mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

$$\mathcal{L}_{r+1}\subset ... \subset \mathcal{L}_{i+1}\subset \mathcal{L}_i \subset \mathcal{L}_{i-1}\subset ... \subset E=\mathcal{L}_1,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that $A\in Hom_{\mathcal{O}_{\mathbb{P}^1}}(E,E^q)$ satisfies the following conditions:

- $A\cdot\mathcal{L}_i\subset\mathcal{L}_{i-1}$,
- $\bar{A}_i:\mathcal{L}_i/\mathcal{L}_{i+1}\to\mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism.

An $(SL(r+1),q)$-*oper* is a $(GL(r+1),q)$-oper with the condition that $det(A)=1$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

QQ-system:

$$\xi_{i+1}Q_i^+(qu)Q_i^-(u)-\xi_iQ_i^+(u)Q_i^-(qu)=\Lambda_i(u)Q_{i-1}^+(u)Q_{i+1}^+(qu)\,,\, i=1,\dots,r$$

$$\xi_1=\frac{1}{\zeta_1},\quad \xi_2=\frac{\zeta_1}{\zeta_2},\quad \dots\quad \xi_r=\frac{\zeta_{r-1}}{\zeta_r},\quad \xi_{r+1}=\frac{1}{\zeta_r},$$

$(SL(r+1), q)$-opers: alternative definition

A $(GL(r+1), q)$-*oper* on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

$$\mathcal{L}_{r+1} \subset ... \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset ... \subset E = \mathcal{L}_1,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that $A \in Hom_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ satisfies the following conditions:

- $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$,
- $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \to \mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism.

An $(SL(r+1), q)$-*oper* is a $(GL(r+1), q)$-oper with the condition that $det(A) = 1$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

QQ-system:

$$\xi_{i+1} Q_i^+(qu) Q_i^-(u) - \xi_i Q_i^+(u) Q_i^-(qu) = \Lambda_i(u) Q_{i-1}^+(u) Q_{i+1}^+(qu)\,, i = 1, \ldots, r$$

$$\xi_1 = \frac{1}{\zeta_1}, \quad \xi_2 = \frac{\zeta_1}{\zeta_2}, \quad \ldots \quad \xi_r = \frac{\zeta_{r-1}}{\zeta_r}, \quad \xi_{r+1} = \frac{1}{\zeta_r},$$

$(SL(r+1),q)$-opers: alternative definition

A $(GL(r+1),q)$-*oper* on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

$$\mathcal{L}_{r+1} \subset ... \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset ... \subset E = \mathcal{L}_1,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that $A \in Hom_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ satisfies the following conditions:

- $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$,
- $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \to \mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism.

An $(SL(r+1),q)$-*oper* is a $(GL(r+1),q)$-oper with the condition that $det(A)=1$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

QQ-system:

$$\xi_{i+1} Q_i^+(qu) Q_i^-(u) - \xi_i Q_i^+(u) Q_i^-(qu) = \Lambda_i(u) Q_{i-1}^+(u) Q_{i+1}^+(qu)\,,\, i=1,\dots,r$$

$$\xi_1 = \frac{1}{\zeta_1}, \quad \xi_2 = \frac{\zeta_1}{\zeta_2}, \quad \dots \quad \xi_r = \frac{\zeta_{r-1}}{\zeta_r}, \quad \xi_{r+1} = \frac{1}{\zeta_r}\,,$$

Bethe coordinates (QQ-system) vs coordinates of line bundle $\mathcal{L}_{r+1}$:

Space of functions on Z-twisted Miura $(SL(r+1), q)$-opers

$$\updownarrow$$

Space of functions on the intersection of two Lagrangian subvarieties in trigonometric Ruijsenaars-Schneider (tRS) phase space.

$$\text{Bethe equations} \leftrightarrow \{H_k(\{\xi_i\}, \{p_i\}, q) = f_k(\{a_i\})\}$$

Here H_k are tRS Hamiltonians, involutive w.r.t. $\Omega = \sum_k \frac{dx_k}{x_k} \wedge \frac{dp_k}{p_k}$:

$$H_k = \sum_{\substack{J\subset\{1,\dots,r+1\} \\ |J|=k}} \prod_{\substack{i\in J \\ j\notin J}} \frac{x_i - qx_j}{x_i - x_j} \prod_{m\in J} p_m$$

and f_k are elementary symmetric functions of a_i.

P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

P. Koroteev, D. Sage, A.Z. '20

tRS integrable system and quantum/classical duality

Bethe coordinates (QQ-system) vs coordinates of line bundle $\mathcal{L}_{r+1}$:

Space of functions on Z-twisted Miura $(SL(r+1), q)$-opers

$$\updownarrow$$

Space of functions on the intersection of two Lagrangian subvarieties in trigonometric Ruijsenaars-Schneider (tRS) phase space.

$$\text{Bethe equations} \leftrightarrow \{H_k(\{\xi_i\}, \{p_i\}, q) = f_k(\{a_i\})\}$$

Here H_k are tRS Hamiltonians, involutive w.r.t. $\Omega = \sum_k \frac{dx_k}{x_k} \wedge \frac{dp_k}{p_k}$:

$$H_k = \sum_{\substack{J\subset\{1,\dots,r+1\} \\ |J|=k}} \prod_{\substack{i\in J \\ j\notin J}} \frac{x_i - qx_j}{x_i - x_j} \prod_{m\in J} p_m$$

and f_k are elementary symmetric functions of a_i.

P. Koroteev, P. Pushkar, A. Smirnov, A.Z. '17

P. Koroteev, D. Sage, A.Z. '20

Duality in many-body formalism

Calogero-Moser phase space: subset of $GL(r+1;\mathbb{C}) \times GL(r+1;\mathbb{C}) \times \mathbb{C}^{r+1} \times \mathbb{C}^{r+1}$:

$$qMT - TM = u \otimes v^T .$$

When M is a diagonal matrix with eigenvalues $\xi_1, \ldots, \xi_{r+1}$, the components of matrix T are:

$$T_{ij} = \frac{u_i v_j}{q\xi_i - \xi_j} .$$

Introducing the momenta: $p_i = -u_i v_i \frac{\prod\limits_{k \neq i} (\xi_i - \xi_k)}{\prod\limits_k (\xi_i - \xi_k q)}$, we obtain:

$$\det\left(z - T(\xi_i, p_i, q)\right) = \sum_{k=0}^{r+1} (-1)^k z^{n-k} H_k(\xi_i, p_i, q) ,$$

Duality:

$$q \mapsto q^{-1}, \qquad M \mapsto T, \qquad T \mapsto M .$$

Or, in other words:

$$q \mapsto q^{-1}, \quad (\{\xi_i\}, \{p_i\}) \to (\{a_i\}, \{\tilde{p}_i\})$$

Duality in many-body formalism

Calogero-Moser phase space: subset of $GL(r+1;\mathbb{C}) \times GL(r+1;\mathbb{C}) \times \mathbb{C}^{r+1} \times \mathbb{C}^{r+1}$:

$$qMT - TM = u \otimes v^T\,.$$

When M is a diagonal matrix with eigenvalues $\xi_1, \ldots, \xi_{r+1}$, the components of matrix T are:

$$T_{ij} = \frac{u_i v_j}{q\xi_i - \xi_j}\,.$$

Introducing the momenta: $p_i = -u_i v_i \frac{\prod\limits_{k\neq i}(\xi_i - \xi_k)}{\prod\limits_{k}(\xi_i - \xi_k q)}$, we obtain:

$$\det\left(z - T(\xi_i, p_i, q)\right) = \sum_{k=0}^{r+1} (-1)^k z^{n-k} H_k(\xi_i, p_i, q)\,,$$

Duality:

$$q \mapsto q^{-1}, \qquad M \mapsto T\,, \qquad T \mapsto M\,.$$

Or, in other words:

$$q \mapsto q^{-1}, \quad (\{\xi_i\}, \{p_i\}) \to (\{a_i\}, \{\tilde{p}_i\})$$

Duality in many-body formalism

Calogero-Moser phase space: subset of $GL(r+1;\mathbb{C}) \times GL(r+1;\mathbb{C}) \times \mathbb{C}^{r+1} \times \mathbb{C}^{r+1}$:

$$qMT - TM = u \otimes v^T .$$

When M is a diagonal matrix with eigenvalues $\xi_1, \dots, \xi_{r+1}$, the components of matrix T are:

$$T_{ij} = \frac{u_i v_j}{q\xi_i - \xi_j} .$$

Introducing the momenta: $p_i = -u_i v_i \frac{\prod_{k \neq i}(\xi_i - \xi_k)}{\prod_k (\xi_i - \xi_k q)}$, we obtain:

$$\det\left(z - T(\xi_i, p_i, q)\right) = \sum_{k=0}^{r+1} (-1)^k z^{n-k} H_k(\xi_i, p_i, q) ,$$

Duality:

$$q \mapsto q^{-1}, \qquad M \mapsto T, \qquad T \mapsto M .$$

Or, in other words:

$$q \mapsto q^{-1}, \quad (\{\xi_i\}, \{p_i\}) \to (\{a_i\}, \{\tilde{p}_i\})$$

Duality in many-body formalism

Calogero-Moser phase space: subset of $GL(r+1;\mathbb{C}) \times GL(r+1;\mathbb{C}) \times \mathbb{C}^{r+1} \times \mathbb{C}^{r+1}$:

$$qMT - TM = u \otimes v^T \,.$$

When M is a diagonal matrix with eigenvalues $\xi_1, \ldots, \xi_{r+1}$, the components of matrix T are:

$$T_{ij} = \frac{u_i v_j}{q\xi_i - \xi_j} \,.$$

Introducing the momenta: $p_i = -u_i v_i \frac{\prod\limits_{k\neq i}(\xi_i - \xi_k)}{\prod\limits_k (\xi_i - \xi_k q)}$, we obtain:

$$\det\left(z - T(\xi_i, p_i, q)\right) = \sum_{k=0}^{r+1} (-1)^k z^{n-k} H_k(\xi_i, p_i, q) \,,$$

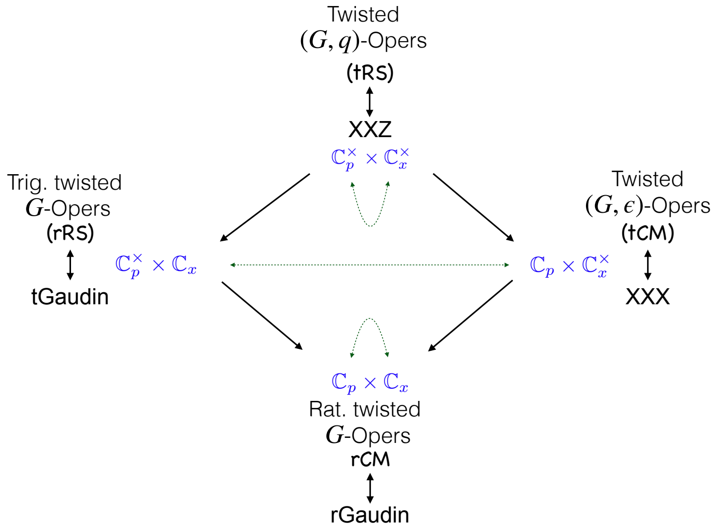
Duality:

$$q \mapsto q^{-1}\,, \qquad M \mapsto T\,, \qquad T \mapsto M\,.$$

Or, in other words:

$$q \mapsto q^{-1}, \quad (\{\xi_i\}, \{p_i\}) \to (\{a_i\}, \{\tilde{p}_i\})$$

Zoo of dualities



Application: 3D mirror symmetry

This is a duality between two types of symplectic resolutions: N, N'.

On enumerative level:

$$K_T^{(q)}(N) \cong K_{T'}^{(q)}(N').$$

T-parameters $\leftrightarrow$ Kähler parameters (parameters of deformation).

For Nakajima quiver varieties in type A:

$$K_T^{(q)}(N) = \text{Bethe algebra}.$$

Notable example: T^*Fl_{r+1}–self-dual.

We used quantum/classical duality to establish 3d mirror symmetry for A-type quiver varieties and used it to prove 3d mirror symmetry for *ADHM* spaces, in particular Hilbert scheme of points:

P. Koroteev, A.M.Z., *3D Mirror symmetry for instanton moduli spaces*, Commun. Math. Phys. 403 (2023) 1005-1068

Application: 3D mirror symmetry

This is a duality between two types of symplectic resolutions: N, N'.

On enumerative level:

$$K_T^{(q)}(N) \cong K_{T'}^{(q)}(N').$$

T-parameters $\leftrightarrow$ Kähler parameters (parameters of deformation).

For Nakajima quiver varieties in type A:

$$K_T^{(q)}(N) = \text{Bethe algebra}.$$

Notable example: $T^* Fl_{r+1}$–self-dual.

We used quantum/classical duality to establish 3d mirror symmetry for A-type quiver varieties and used it to prove 3d mirror symmetry for *ADHM* spaces, in particular Hilbert scheme of points:

P. Koroteev, A.M.Z., *3D Mirror symmetry for instanton moduli spaces*, Commun. Math. Phys. 403 (2023) 1005-1068

Application: 3D mirror symmetry

This is a duality between two types of symplectic resolutions: N, N'.

On enumerative level:

$$K_T^{(q)}(N) \cong K_{T'}^{(q)}(N').$$

T-parameters $\leftrightarrow$ Kähler parameters (parameters of deformation).

For Nakajima quiver varieties in type A:

$$K_T^{(q)}(N) = \text{Bethe algebra}.$$

Notable example: T^*Fl_{r+1}–self-dual.

We used quantum/classical duality to establish 3d mirror symmetry for A-type quiver varieties and used it to prove 3d mirror symmetry for *ADHM* spaces, in particular Hilbert scheme of points:

P. Koroteev, A.M.Z., *3D Mirror symmetry for instanton moduli spaces*, Commun. Math. Phys. 403 (2023) 1005-1068

Application: 3D mirror symmetry

This is a duality between two types of symplectic resolutions: N, N'.

On enumerative level:

$$K_T^{(q)}(N) \cong K_{T'}^{(q)}(N').$$

T-parameters $\leftrightarrow$ Kähler parameters (parameters of deformation).

For Nakajima quiver varieties in type A:

$$K_T^{(q)}(N) = \text{Bethe algebra}.$$

Notable example: T^*Fl_{r+1}–self-dual.

We used quantum/classical duality to establish 3d mirror symmetry for A-type quiver varieties and used it to prove 3d mirror symmetry for $ADHM$ spaces, in particular Hilbert scheme of points:

P. Koroteev, A.M.Z., *3D Mirror symmetry for instanton moduli spaces*, Commun. Math. Phys. 403 (2023) 1005-1068

Thank you!