Homotopy Gerstenhaber algebras, Courant algebroids, and Field Equations

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Outline

Sigma-models and conformal invariance conditions

Beltrami-Courant differential and first order sigma-models

Vertex/Courant algebroids, G_∞-algebras and quasiclassical limit

Einstein Equations from G_∞-algebras

Sigma-models and conformal invariance conditions

Sigma-models for string theory in curved spacetimes:

Let $X : \Sigma \to M$, where Σ is a compact Riemann surface (worldsheet) and M is a Riemannian manifold (target space).

Action functional of sigma model:

$$S_{so} = \frac{1}{4\pi h} \int_\Sigma (G_{\mu\nu}(X) dX^\mu \wedge * dX^\nu + X^* B)$$

where G is a metric on M, B is a 2-form on M.

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On the quantum level one can add one more term to the action (due to E. Fradkin and A. Tseytlin):

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where function Φ is called *dilaton*, γ is a metric on Σ.

In order to make sense of path integral

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Conformal invariance conditions

$$\mu\frac{d}{d\mu}G_{\mu\nu}=\beta^{G}_{\mu\nu}(G,B,\Phi,h)=0,\quad \mu\frac{d}{d\mu}B_{\mu\nu}=\beta^{B}_{\mu\nu}(G,B,\Phi,h)=0,$$
$$\mu\frac{d}{d\mu}\Phi=\beta^{\Phi}(G,B,\Phi,h)=0$$

at the level h^0 turn out to be Einstein Equations with 2-form field B and dilaton Φ:

$$R_{\mu\nu}=\frac{1}{4}H_{\mu}^{\lambda\rho}H_{\nu\lambda\rho}-2\nabla_{\mu}\nabla_{\nu}\Phi,$$
$$\nabla^{\mu}H_{\mu\nu\rho}-2(\nabla^{\lambda}\Phi)H_{\lambda\nu\rho}=0,$$
$$4(\nabla_{\mu}\Phi)^2-4\nabla_{\mu}\nabla^{\mu}\Phi+R+\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}=0,$$

where 3-form $H=dB$, and $R_{\mu\nu}, R$ are Ricci and scalar curvature correspondingly.

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Linearized Einstein Equations and their symmetries:
$(G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu},\ B_{\mu\nu} = b_{\mu\nu},\ \Phi = \phi)$:

$$Q^\eta \Psi(s,b,\phi) = 0, \quad \Psi^s(s,b,\phi) \to \Psi(s,b,\phi) + Q^\eta \Lambda$$

in a semi-infinite complex associated to Virasoro module of Hilbert space of states for the "free" theory, asssociated to flat metric.

It was conjectured (A. Sen, B. Zwiebach,...) in the early 90s that Einstein equations with h-corrections are Generalized Maurer-Cartan (GMC) Equations:

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i) Introducing complex structure:

Proper chiral "free action" $\to$ sheaves of vertex algebras/vertex algebroids.

Metric, B-field $\to$ Beltrami-Courant differential.

ii) Vertex algebroids $\to$ G_∞-algebras (G stands for Gerstenhaber).

Quasiclassical limit:
vertex algebroid $\to$ Courant algebroid, G_∞ algebra is truncated.

iii) Einstein equations and their h-corrections via Generalized Maurer-Cartan equation for L_∞-subalgebra of $G_\infty \otimes \bar{G}_\infty$.

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First order version of sigma-model action

We start from the action functional:

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma \mathcal{L}_0, \quad \mathcal{L}_0 = \langle p \wedge \bar{\partial} X\rangle - \langle \bar{p} \wedge \partial X\rangle,$$

where p, $\bar{p}$ are sections of $X^*(\Omega^{(1,0)}(M)) \otimes \Omega^{(1,0)}(\Sigma)$, $X^*(\Omega^{(0,1)}(M)) \otimes \Omega^{(0,1)}(\Sigma)$ correspondingly.

Infinitesimal local symmetries:

$$\mathcal{L}_0 \to \mathcal{L}_0 + d\xi$$

For holomorphic transformations we have:

$$X^i \to X^i - v^i(X), X^{\bar{i}} \to X^{\bar{i}} - v^{\bar{i}}(\bar{X}),$$
$$p_i \to p_i + \partial_i v^k p_k, \quad p_{\bar{i}} \to p_{\bar{i}} + \partial_{\bar{i}} v^{\bar{k}} p_{\bar{k}}$$
$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k), \quad p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}).$$

Not invariant under general diffeomorphisms, i.e.

$$\delta\mathcal{L}_0 = -\langle \bar{\partial} v, p \wedge \bar{\partial} X\rangle + \langle \partial \bar{v}, \bar{p} \wedge \partial X\rangle.$$

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It is necessary to add extra terms:

$$\delta\mathcal{L}_\mu = -\langle \mu, p \wedge \bar\partial X\rangle - \langle \bar\mu, \partial X \wedge \bar p\rangle,$$

where $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}(M))$, $\bar\mu \in \Gamma(T^{(0,1)}M \otimes T^{*(1,0)}(M))$, so that: $\mu \to \mu - \bar\partial v + \dots$, $\bar\mu \to \bar\mu - \partial \bar v + \dots$.

Continuing the procedure:

$$\tilde{\mathcal{L}} = \langle p \wedge \bar\partial X\rangle - \langle \bar p \wedge \partial X\rangle - \langle \mu, p \wedge \bar\partial X\rangle - \langle \bar\mu, \partial X \wedge \bar p\rangle - \langle b, \partial X \wedge \bar\partial X\rangle,$$

where

$$\mu^i_{\bar j} \to \mu^i_{\bar j} - \partial_{\bar j} v^i + v^k\partial_k \mu^i_{\bar j} + v^{\bar k}\partial_{\bar k}\mu^i_{\bar j} + \mu^i_{\bar k}\partial_{\bar j} v^{\bar k} - \mu^k_{\bar j}\partial_k v^i + \mu^i_{\bar l}\mu^k_{\bar j}\partial_k v^{\bar l},$$

$$b_{i\bar j} \to b_{i\bar j} + v^k\partial_k b_{i\bar j} + v^{\bar k}\partial_{\bar k} b_{i\bar j} + b_{i\bar k}\partial_{\bar j} v^{\bar k} + b_{l\bar j}\partial_i v^l + b_{i\bar k}\mu^k_{\bar j}\partial_k v^{\bar k} + b_{l\bar j}\bar\mu^{\bar k}_i \partial_{\bar k} v^l,$$

so that the transformations of X- and p- fields are:

$$X^i \to X^i - v^i(X,\bar X), \quad p_i \to p_i + p_k\partial_i v^k - p_k \mu^k_{\bar l}\partial_i v^{\bar l} - b_{j\bar k}\partial_i v^{\bar k}\partial X^j,$$

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Similarly, for the 1-form transformation we obtain:

$$b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu^i_{\bar{j}}(\partial_i\omega_k - \partial_k\omega_i) +$$
$$\bar{\mu}^{\bar{s}}_i(\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}^{\bar{i}}_j\mu^s_{\bar{k}}(\partial_s\omega_{\bar{i}} - \partial_{\bar{i}}\omega_s)$$

and

$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_k\partial_i\omega_{\bar{s}}\partial X^k,$$
$$p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial} X^r - \mu^s_{\bar{k}}\partial_i\omega_s\bar{\partial} X^{\bar{k}}.$$

For simplicity:

$$E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}},$$
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$$E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}},$$
$$\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.$$

Let $\tilde{\mathbb{M}} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$\tilde{\mathbb{M}} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

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Let us describe ϕ_1, ϕ_2 algebraically. In order to do that we need to pass to jet bundles, i.e.

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One can write formally:

$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$
$$\tilde{\mathbb{M}} = \sum_I a^I \otimes \bar{a}^I,$$

where $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$, $f^I \in J^\infty(\mathcal{O}_M)$ and $\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$, $\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)$. Then

$$\phi_1(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where $[\cdot,\cdot]_D$ is a *Dorfman bracket*:

$$[v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v\omega,$$
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Courant bracket is the antisymmetrized version of $[\cdot,\cdot]_D$.
Similarly:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}}$$
$$\frac{1}{2}\sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J (\bar{f}^I) \bar{a}^K + \frac{1}{2}\sum_{I,J,K} a^J (f^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

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Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on M) via sheaves of VOA on M (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set U of M we have VOA:

$$V=\sum_{n=0}^{\infty} V_n, \quad Y: V \to End(V)[[z,z^{-1}]],$$

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$F(U)$ generated by X^i_0-modes.

Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on M) via sheaves of VOA on M (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).
On the open set U of M we have VOA:

$$V=\sum_{n=0}^{\infty} V_n, \quad Y: V \to End(V)[[z,z^{-1}]],$$

generated by:

$$[X^i(z),p_j(w)]=h\delta^i_j\delta(z-w), \quad i,j=1,2,\dots,D/2$$
$$X^i(z)=\sum_{r\in\mathbb{Z}} X^i_r z^{-r}, p_j(z)=\sum_{s\in\mathbb{Z}} p_{j,s}z^{-s-1}\in End(V)[[z,z^{-1}]],$$

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The Virasoro element is:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{h} : \langle p(z)\partial X(z)\rangle : + \partial^2\phi'(X(z)).$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}(n^3 - n)\delta_{n,-m}$$

corresponding to correction:

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where $\phi' = \log\Omega$, where $\Omega(X)dX^1 \wedge \cdots \wedge dX^n$ is a holomorphic volume form, i.e. for globally defined $T(z)$, M has to be Calabi-Yau.
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V can be reproduced from V_0 and V_1 as a *vertex envelope*. The structure of vertex algebra imposes algebraic relations on $V_0 \oplus V_1$ giving it a structure of a *vertex algebroid*.

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A *vertex* $\mathcal{O}_M$*-algebroid* is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with

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Vertex algebra V is a Virasoro module. The corresponding semi-infinite complex V^{semi} (the analogue of Chevalley complex for Virasoro algebra) is a vertex algebra too:

$$V^{semi} = V \otimes \Lambda,$$
$$\Lambda \quad \text{generated} \quad \text{by} \quad [b(z), c(w)]_+ = \delta(z-w).$$

The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1} = c(z)T(z) + :c(z)\partial c(z) b(z):$$

is nilpotent when $D = 26$ (famous dimension 26!). However, we will consider subcomplex of light modes (i.e. $L_0 = 0$) denoted in the following as $(\mathcal{F}^{\cdot}_h, Q)$, where we can drop this condition:

$$\begin{tikzcd}
& \mathcal{V}^h \arrow[drr, "-\frac{1}{2}h\mathrm{div}"] \arrow[d, phantom, "\bigoplus"] & \mathcal{V}^h \arrow[dr, "\frac{1}{2}h\mathrm{div}"] \arrow[d, phantom, "\bigoplus"] & \\
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Vertex algebra V is a Virasoro module. The corresponding semi-infinite complex V^{semi} (the analogue of Chevalley complex for Virasoro algebra) is a vertex algebra too:

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The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

$$(A,B)_h = Res_z \frac{A(z)B}{z}$$

$$Q(a_1,a_2)_h = (Qa_1,a_2)_h + (-1)^{|a_1|}(a_1,Qa_2)_h,$$
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Operator $\mathbf{b}$ of degree -1 (0-mode of $b(z)$) on $(\mathcal{F}_h^{\cdot}, Q)$ which anticommutes with Q:

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One can define a bracket:

$$(-1)^{|a_1|}\{a_1, a_2\}_h = \mathbf{b}(a_1, a_2)_h - (\mathbf{b}a_1, a_2)_h - (-1)^{|a_1|}(a_1\mathbf{b}a_2)_h,$$

so that together with Q, $(\cdot,\cdot)_h$ it satisfies the relations of homotopy Gerstenhaber algebra:

$$\{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\}_h =$$
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The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to G_∞-algebra.

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Let A be a graded vector space, consider free graded Lie algebra $Lie(A)$.

$$Lie^{k+1}(A) = [A, Lie^k A], \quad Lie^1(A) = A.$$

Consider free graded commutative algebra GA on the suspension $(Lie(A))[-1]$, i.e.

$$GA = \oplus_n \bigwedge^n Lie(A)[-n]$$

There are natural $[\cdot,\cdot]$, $\wedge$ operations on GA of degree -1, 0 correpondingly, generating a Gerstenhaber algebra.

A G_∞-*algebra* (Tamarkin, Tsygan, 2000) is a graded space V with a differential ∂ of degree 1 of $G(V[1]^*)$, such that ∂ is a derivation w.r.t bracket and the product.

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Homotopy algebras: G_∞, L_∞, C_∞

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Restriction of ∂ on $V[1]^*$:

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Quasiclassical limit of LZ G_∞ algebra

The following complex $(\mathcal{F}^{\cdot}, Q)$:

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 & \bigoplus & \bigoplus & \\
\mathcal{O}_M \arrow[uur, "\partial"] & h\mathcal{O}_M \arrow[uur, "\partial"] \arrow[r, "id"'] & h\mathcal{O}_M & h^2\mathcal{O}_M
\end{tikzcd}
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is a subcomplex of $(\mathcal{F}_h^{\cdot}, Q)$. Then

$$(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\}_h : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h],$$
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The symmetrized operations $(\cdot,\cdot)_0$, $\{\cdot,\cdot\}_0, \ldots$ satisfy the relations of the homotopy Gerstenhaber algebra, so that all non-covariant higher-order terms disappear from the multilinear operations.

The resulting C_∞ and L_∞ algebras are reduced to C_3 and L_3 algebras.

A.Z., Comm. Math. Phys. 303 (2011) 331-359.

Conjecture: This G_∞-algebra is the G_3-algebra (no homotopies beyond trilinear operations).

Classical limit procedure for vertex algebroid (due to P. Bressler):
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A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with a structure of a Leibniz $\mathbb{C}$-algebra $[\cdot,\cdot]_0 : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \to \mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0 : \mathcal{Q} \to \Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{Q} \to \mathcal{O}_M$, a derivation $\partial : \mathcal{O}_M \to \mathcal{Q}$ which satisfy

$$\pi \circ \partial = 0, \quad [q_1, fq_2]_0 = f[q_1, q_2]_0 + \pi_0(q_1)(f)q_2$$
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We show that it is a part of a more general structure, homotopy Gerstenhaber algebra.

Question: Is there a direct path (avoiding vertex algebra) from Courant algebroid to G_3-algebra? Odd analogue of Manin double?

Remark. C_3-algebra is related to gauge theory. The appropriate "metric" deformation gives a Yang-Mills C_3-algebra on a flat space.

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Simplest version: $G_\infty \to$ Gerstenhaber algebra

Subcomplex $(\mathcal{F}^{\cdot}_{sm}, Q)$:

$$\begin{tikzcd}
 & \mathcal{O}(T^{(1,0)}M) \arrow[dddrr, "-\frac{1}{2}\mathrm{div}"] & & \mathcal{O}(T^{(1,0)}M) \arrow[dddr, "\frac{1}{2}\mathrm{div}"] & \\
 & & & 0 & \\
 & \bigoplus & & \bigoplus & \\
\mathbb{C} \arrow[uuur, "0"] & & \mathbb{C} \arrow[uuur] \arrow[r, "i"'] & \mathcal{O}_M & \mathcal{O}_M
\end{tikzcd}$$

The G_∞ algebra degenerates to G-algebra. Moreover, due to $\mathbf{b}_0$ it is a BV-algebra. Combine chiral and antichiral part:

$$\mathbf{F}^{\cdot}_{sm} = \mathcal{F}^{\cdot}_{sm} \otimes \bar{\mathcal{F}}^{\cdot}_{sm}$$

$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^-(a_1, a_2) - (\mathbf{b}^- a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^- a_2),$$

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\mathbb{C} \arrow[uur, "0"] & \mathbb{C} \arrow[uur, "0"] \arrow[r, "i"'] & \mathcal{O}_M & \mathcal{O}_M
\end{tikzcd}$$

The G_∞ algebra degenerates to G-algebra. Moreover, due to $\mathbf{b}_0$ it is a BV-algebra. Combine chiral and antichiral part:

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$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^-(a_1, a_2) - (\mathbf{b}^- a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^- a_2),$$

where $\mathbf{b}^- = \mathbf{b} - \bar{\mathbf{b}}$.

Simplest version: $G_\infty \to$ Gerstenhaber algebra

Subcomplex $(\mathcal{F}^{\cdot}_{sm}, Q)$:

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Components:$(g, \bar{v}, v, \phi, \bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

1). Vector field $div_\Omega g$, where $\log\Omega = -2\Phi_0 = -2(\phi' + \bar{\phi}' + \phi + \bar{\phi})$ and $\partial_i\partial_{\bar{j}}\Phi_0 = 0$, is such that its $\Gamma(T^{(1,0)}M)$, $\Gamma(T^{(0,1)}M)$ components are correspondingly holomorphic and antiholomorphic.
2). Bivector field $g\in\Gamma(T^{(1,0)}M\otimes T^{(0,1)}M)$ obeys the following equation:

$$[[g,g]] + \mathcal{L}_{div_\Omega(g)}g = 0,$$

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These are Einstein equations with the following constraints:

$$G_{i\bar k} = g_{i\bar k}, \quad B_{i\bar k} = -g_{i\bar k}, \quad \Phi = \log\sqrt{g} + \Phi_0,$$
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Physically:

$$\int [dp][d\bar p][dX][d\bar X]e^{-\frac{1}{2\pi i h}\int_\Sigma(\langle p\wedge\bar\partial X\rangle - \langle\bar p\wedge\partial X\rangle - \langle g, p\wedge\bar p\rangle) + \int_\Sigma R^{(2)}(\gamma)\Phi_0(X)} =$$
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Main Conjecture

Consider

$$\mathbf{F}^{\cdot}_{b^-} = \mathcal{F}^{\cdot} \otimes \bar{\mathcal{F}}^{\cdot}|_{b^-=0}$$

with the L_∞-algebra structure given by Lian-Zuckerman construction.

One can explicitly check that GMC symmetry
$(\Psi = \Psi(\mathbb{M}, \Phi, \mathrm{auxiliary\ \ fields})$

$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \ldots,$$

reproduces

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Thank you!