Einstein field equations, Courant algebroids and Homotopy algebras

Anton M. Zeitlin

Columbia University, Department of Mathematics

University of Toronto
Toronto
Septemeber 27, 2016
Outline

Sigma-models and conformal invariance conditions

Beltrami-Courant differential and first order sigma-models

Vertex/Courant algebroids, \( G_\infty \)-algebras and quasiclassical limit

Einstein Equations from \( G_\infty \)-algebras
Sigma-models for string theory in curved spacetimes:

Let $X : \Sigma \rightarrow M$, where $\Sigma$ is a compact Riemann surface (worldsheet) and $M$ is a Riemannian manifold (target space).

Action functional of sigma model:

$$S_{so} = \frac{1}{4\pi\hbar} \int_\Sigma (G_{\mu\nu}(X) dX^\mu \wedge *dX^\nu + X^* B)$$

where $G$ is a metric on $M$, $B$ is a 2-form on $M$.

Symmetries:

i) conformal symmetry on the worldsheet,

ii) diffeomorphism symmetry and $B \rightarrow B + d\lambda$ on target space.
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On the quantum level one can add one more term to the action (due to E. Fradkin and A. Tseytlin):

\[ S_{so} \rightarrow S_{so}^{\Phi} = S_{so} + \int_{\Sigma} \Phi(X)R^{(2)}(\gamma)\text{vol}_\Sigma, \]

where function \( \Phi \) is called \textit{dilaton}, \( \gamma \) is a metric on \( \Sigma \).

In order to make sense of path integral

\[ Z = \int DX \ e^{-S_{so}^{\Phi}(X, \gamma)} \]

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Conformal invariance conditions

\[ \mu \frac{d}{d\mu} G_{\mu\nu} = \beta^G_{\mu\nu}(G, B, \Phi, h) = 0, \quad \mu \frac{d}{d\mu} B_{\mu\nu} = \beta^B_{\mu\nu}(G, B, \Phi, h) = 0, \]

\[ \mu \frac{d}{d\mu} \Phi = \beta^\Phi(G, B, \Phi, h) = 0 \]

At the level \( h^0 \) turn out to be Einstein Equations with 2-form field \( B \) and dilaton \( \Phi \):

\[ R_{\mu\nu} = \frac{1}{4} H^\lambda_{\mu\rho} H_{\nu\lambda\rho} - 2 \nabla_\mu \nabla_\nu \Phi, \]

\[ \nabla^\mu H_{\mu\nu\rho} - 2(\nabla^\lambda \Phi) H_{\lambda\nu\rho} = 0, \]

\[ 4(\nabla_\mu \Phi)^2 - 4 \nabla_\mu \nabla^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0, \]

Where 3-form \( H = dB \), and \( R_{\mu\nu}, R \) are Ricci and scalar curvature correspondingly.
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In the early days of string theory:

Linearized Einstein Equations and their symmetries:

\[ (G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu}, \ B_{\mu\nu} = b_{\mu\nu}, \ \Phi = \phi): \]

\[ Q^n \psi(s, b, \phi) = 0, \quad \psi^s(s, b, \phi) \to \psi(s, b, \phi) + Q^n \Lambda \]

in a semi-infinite complex associated to Virasoro module of Hilbert space of states for the "free" theory, associated to flat metric.

It was conjectured (A. Sen, B. Zwiebach,...) in the early 90s that Einstein equations with \( h \)-corrections are Generalized Maurer-Cartan (GMC) Equations:

\[ Q^n \psi + \frac{1}{2} [\psi, \psi]_h + \frac{1}{3!} [\psi, \psi, \psi]_h + ... = 0 \]

\[ \psi \to \psi + Q^n \Lambda + [\psi, \Lambda]_h + \frac{1}{2} [\psi, \psi, \Lambda]_h + ... , \]

where \([\cdot, \cdot, ..., \cdot]_h \) operations, together with differential \( Q^n \) satisfy certain bilinear relations and generate \( L_\infty \)-algebra (\( L \) stands for Lie).
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In this talk:


ii) Vertex algebroids → $G_\infty$-algebras ($G$ stands for Gerstenhaber). Quasiclassical limit: vertex algebroid → Courant algebroid, $G_\infty$ algebra is truncated.

iii) Einstein equations and their $\hbar$-corrections via Generalized Maurer-Cartan equation for $L_\infty$-subalgebra of $G_\infty \otimes \tilde{G}_\infty$. 
In this talk:

i) Introducing complex structure:
Proper chiral "free action" → sheaves of vertex algebras/vertex algebroids.
Metric, $B$-field → Beltrami-Courant differential.

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First order version of sigma-model action

We start from the action functional:

\[ S_0 = \frac{1}{2\pi i\hbar} \int_{\Sigma} \mathcal{L}_0, \quad \mathcal{L}_0 = \langle p \wedge \bar{\partial}X \rangle - \langle \bar{p} \wedge \partial X \rangle, \]

where \( p, \bar{p} \) are sections of \( X^* (\Omega^{(1,0)}(M)) \otimes \Omega^{(1,0)}(\Sigma) \), \( X^* (\Omega^{(0,1)}(M)) \otimes \Omega^{(0,1)}(\Sigma) \) correspondingly.

Infinitesimal local symmetries:

\[ \mathcal{L}_0 \to \mathcal{L}_0 + d\xi \]

For holomorphic transformations we have:

\[ X^i \to X^i - \nu^i (X), \quad X^\bar{i} \to X^\bar{i} - \bar{\nu}\bar{i} (\bar{X}), \]
\[ p_i \to p_i + \partial_i \nu^k p_k, \quad p_i \to p_i + \partial_i \bar{\nu}^k \bar{p}_k, \]
\[ p_i \to p_i - \partial X^k (\partial_k \omega_i - \partial_i \omega_k), \quad p_i \to p_i - \bar{\partial} X^\bar{k} (\partial_{\bar{k}} \omega_i - \partial_i \omega_{\bar{k}}). \]

Not invariant under general diffeomorphisms, i.e.

\[ \delta \mathcal{L}_0 = -\langle \bar{\partial} \nu, p \wedge \bar{\partial}X \rangle + \langle \partial \bar{\nu}, \bar{p} \wedge \partial X \rangle. \]
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\delta \mathcal{L}_0 = -\langle \bar{\partial} \nu, p \wedge \bar{\partial} X \rangle + \langle \partial \bar{\nu}, \bar{p} \wedge \partial X \rangle.
\]
It is necessary to add extra terms:

\[ \delta \mathcal{L}_\mu = -\langle \mu, p \wedge \bar{\partial}X \rangle - \langle \bar{\mu}, \partial X \wedge \bar{p} \rangle, \]

where \( \mu \in \Gamma(\mathcal{T}^{(1,0)} M \otimes T^*_{(0,1)}(M)) \), \( \bar{\mu} \in \Gamma(\mathcal{T}^{(0,1)} M \otimes T^*_{(1,0)}(M)) \), so that: \( \mu \rightarrow \mu - \bar{\partial}v + \ldots, \bar{\mu} \rightarrow \bar{\mu} - \partial \bar{v} + \ldots \).

Continuing the procedure:

\[ \tilde{\mathcal{L}} = \langle p \wedge \bar{\partial}X \rangle - \langle \bar{p} \wedge \partial X \rangle - \langle \mu, p \wedge \bar{\partial}X \rangle - \langle \bar{\mu}, \partial X \wedge \bar{p} \rangle - \langle b, \partial X \wedge \bar{\partial}X \rangle, \]

where

\[
\begin{align*}
\mu^i_j & \rightarrow \\
\mu^i_j - \partial^j_i v^i + v^k \partial_k \mu^i_j + v^k \partial_k \mu^i_j + \mu^i_k \partial_j v^k - \mu^i_j \partial_k v^i + \mu^i_j \mu^k_j \partial_k \bar{v}^l,
\end{align*}
\]

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b_{ij} & \rightarrow \\
b_{ij} + v^k \partial_k b_{ij} + v^k \partial_k b_{ij} + b_{ik} \partial_j v^k + b_{ij} \partial_i v^l + b_{ik} \mu^k_j \partial_k v^k + b_{ij} \bar{\mu}^k_j \partial_k v^l,
\end{align*}
\]

so that the transformations of \( X \)- and \( p \)- fields are:

\[
\begin{align*}
X^i & \rightarrow X^i - v^i(X, \bar{X}), & p_i & \rightarrow p_i + p_k \partial_i v^k - p_k \mu^k_j \partial_i \bar{v}^l - b_{jk} \partial_i \bar{v}^l \partial X^j, \\
X^{\bar{i}} & \rightarrow X^{\bar{i}} - v^{\bar{i}}(X, \bar{X}), & \bar{p}_{\bar{i}} & \rightarrow \bar{p}_{\bar{i}} + \bar{p}_{\bar{k}} \partial_{\bar{i}} v^{\bar{k}} - \bar{p}_{\bar{k}} \bar{\mu}^k_{\bar{i}} \partial_{\bar{i}} v^l - b_{jk} \partial_{\bar{i}} v^{\bar{k}} \partial X^j.
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where

$$\mu^i_j \to$$

$$\mu^i_j - \partial^i_j v^i + v^k \partial_k \mu^i_j + \bar{v}^k \partial_k \bar{\mu}^i_j + \mu^i_k \partial^j \bar{v}^k - \mu^i_j \partial^k v^i + \mu^i_j \mu^k_l \partial^i \bar{v}^l,$$

$$b_{ij} \to$$

$$b_{ij} + v^k \partial_k b_{ij} + \bar{v}^k \partial_k \bar{b}_{ij} + b_{ik} \partial^j \bar{v}^k + b_{ij} \partial^i v^l + b_{ik} \mu^k_l \partial^i \bar{v}^l + b_{ij} \bar{\mu}^k_l \partial^k \bar{v}^l,$$

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$$X^i \to X^i - v^i(X, \bar{X}), \quad p_i \to p_i + p_k \partial_i v^k - p_k \mu^k_l \partial_i \bar{v}^l - b_{jk} \partial_i \bar{v}^k \partial X^j,$$

$$X^\bar{i} \to X^\bar{i} - \bar{v}^{\bar{i}}(X, \bar{X}), \quad \bar{p}_{\bar{i}} \to \bar{p}_{\bar{i}} + \bar{p}_{\bar{k}} \partial_{\bar{i}} \bar{v}^{\bar{k}} - \bar{p}_{\bar{k}} \bar{\mu}^{\bar{k}}_{\bar{l}} \partial_{\bar{i}} v^{\bar{l}} - b_{\bar{j}k} \partial_{\bar{i}} v^{\bar{k}} \partial X^j.$$
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Continuing the procedure:

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so that the transformations of $X$- and $p$- fields are:

$$X^i \rightarrow X^i - v^i(X, \bar{X}), \quad p_i \rightarrow p_i + p_k \partial_i v^k - p_k \mu^k_i \partial_i v^l - b_{j\bar{k}} \partial_i v^k \partial X^j,$$

$$\bar{X}^i \rightarrow \bar{X}^i - \bar{v}^i(X, \bar{X}), \quad \bar{p}_i \rightarrow \bar{p}_i + \bar{p}_k \partial_i \bar{v}^k - \bar{p}_k \bar{\mu}^k_i \partial_i v^l - b_{j\bar{k}} \partial_i v^k \partial \bar{X}^j.$$
Similarly, for the 1-form transformation we obtain:

\[ b_{ij} \rightarrow b_{ij} + \partial_j \omega_i - \partial_i \omega_j + \mu^i_j (\partial_i \omega_k - \partial_k \omega_i) + \]

\[ \bar{\mu}_i^\bar{s} (\partial_j \omega_{\bar{s}} - \partial_{\bar{s}} \omega_j) + \bar{\mu}_j^i \mu_k^s (\partial_s \omega_i - \partial_i \omega_s) \]

and

\[ p_i \rightarrow p_i - \partial X^k (\partial_k \omega_i - \partial_i \omega_k) - \partial \omega_i \partial X^r - \bar{\mu}_k^i \partial_i \omega_{\bar{s}} \partial X^k, \]

\[ p_{\bar{i}} \rightarrow p_{\bar{i}} - \bar{\partial} X^\bar{k} (\partial_{\bar{k}} \omega_{\bar{i}} - \partial_{\bar{i}} \omega_{\bar{k}}) - \partial \omega_{\bar{i}} \bar{\partial} X^r - \mu_k^s \partial_i \omega_s \bar{\partial} X^\bar{k}. \]

For simplicity:

\[ E = TM \oplus T^* M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}}, \]

\[ \mathcal{E} = T^{(1,0)} M \oplus T^{* (1,0)} M, \quad \bar{\mathcal{E}} = T^{(0,1)} M \oplus T^{* (0,1)} M. \]
Similarly, for the 1-form transformation we obtain:

\[ b_{i\bar{j}} \rightarrow b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu^i_j (\partial_i\omega_k - \partial_k\omega_i) + \]
\[ \bar{\mu}_{i}^{\bar{s}} (\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}_{j}^{\bar{s}} \mu_k^s (\partial_s\omega_i - \partial_i\omega_s) \]

and

\[ p_i \rightarrow p_i - \partial X^k (\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i \partial X^{\bar{r}} - \bar{\mu}_{k}^{\bar{s}} \partial_i\omega_{\bar{s}} \partial X^k, \]
\[ p_{\bar{i}} \rightarrow p_{\bar{i}} - \bar{\partial} X^k (\partial_k\omega_{\bar{i}} - \partial_{\bar{i}}\omega_k) - \partial_r\omega_{\bar{i}} \bar{\partial} X^r - \mu_k^s \partial_i\omega_s \bar{\partial} X^k. \]

For simplicity:

\[ E = TM \oplus T^* M, \quad \bar{E} = \mathcal{E} \oplus \bar{\mathcal{E}}, \]
\[ \mathcal{E} = T^{(1,0)} M \oplus T^{* (1,0)} M, \quad \bar{\mathcal{E}} = T^{(0,1)} M \oplus T^{* (0,1)} M. \]
Let \( \tilde{M} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}}) \), such that

\[
\tilde{M} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.
\]

Introduce \( \alpha \in \Gamma(E) \), i.e. \( \alpha = (v, \bar{v}, \omega, \bar{\omega}) \). Let \( D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}}) \), such that

\[
D\alpha = \begin{pmatrix} 0 & \partial v \\ \partial \bar{v} & \partial \bar{\omega} - \partial \omega \end{pmatrix}.
\]

Then the transformation of \( \tilde{M} \) is:

\[
\tilde{M} \to \tilde{M} - D\alpha + \phi_1(\alpha, \tilde{M}) + \phi_2(\alpha, \tilde{M}, \tilde{M}).
\]

Let us describe \( \phi_1, \phi_2 \) algebraically. In order to do that we need to pass to jet bundles, i.e.

\[
\alpha \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}_M),
\]

\[
\tilde{M} \in J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})).
\]
Let $\tilde{M} \in \Gamma(\mathcal{E} \otimes \tilde{\mathcal{E}})$, such that

$$\tilde{M} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$ 

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \tilde{\mathcal{E}})$, such that

$$D\alpha = \begin{pmatrix} 0 & \bar{\partial}v \\ \partial \bar{v} & \partial \bar{\omega} - \bar{\partial}\omega \end{pmatrix}.$$ 

Then the transformation of $\tilde{M}$ is:

$$\tilde{M} \to \tilde{M} - D\alpha + \phi_1(\alpha, \tilde{M}) + \phi_2(\alpha, \tilde{M}, \tilde{M}).$$ 

Let us describe $\phi_1, \phi_2$ algebraically. In order to do that we need to pass to jet bundles, i.e.

$$\alpha \in J^\infty(\Omega_M) \otimes J^\infty(\bar{\Omega}(\tilde{\mathcal{E}})) \oplus J^\infty(\Omega(\mathcal{E})) \otimes J^\infty(\bar{\Omega}_M),$$

$$\tilde{M} \in J^\infty(\Omega(\mathcal{E})) \otimes J^\infty(\bar{\Omega}(\tilde{\mathcal{E}}))$$
Let $\tilde{M} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$\tilde{M} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$ 

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$D\alpha = \begin{pmatrix} 0 & \bar{\partial}v \\ \partial \bar{v} & \partial \bar{\omega} - \bar{\partial}\omega \end{pmatrix}.$$ 

Then the transformation of $\tilde{M}$ is:

$$\tilde{M} \rightarrow \tilde{M} - D\alpha + \phi_1(\alpha,\tilde{M}) + \phi_2(\alpha,\tilde{M},\tilde{M}).$$

Let us describe $\phi_1, \phi_2$ algebraically. In order to do that we need to pass to jet bundles, i.e.

$$\alpha \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\mathcal{O}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\mathcal{O}_M),$$

$$\tilde{M} \in J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\mathcal{O}(\bar{\mathcal{E}}))$$
Let $\tilde{M} \in \Gamma(\mathcal{E} \otimes \tilde{\mathcal{E}})$, such that

$$\tilde{M} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$ 

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \tilde{\mathcal{E}})$, such that

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Then the transformation of $\tilde{M}$ is:

$$\tilde{M} \to \tilde{M} - D\alpha + \phi_1(\alpha, \tilde{M}) + \phi_2(\alpha, \tilde{M}, \tilde{M}).$$

Let us describe $\phi_1, \phi_2$ algebraically. In order to do that we need to pass to jet bundles, i.e.

$$\alpha \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O} \mathcal{E}) \otimes J^\infty(\bar{\mathcal{O}}_M),$$

and

$$\tilde{M} \in J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})).$$
One can write formally:

\[ \alpha = \sum_J f^J \otimes \tilde{b}^J + \sum_K b^K \otimes \tilde{f}^K, \]

\[ \tilde{\mathcal{M}} = \sum_I a^I \otimes \bar{a}^I, \]

where \( a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E})), f^I \in J^\infty(\mathcal{O}_M) \) and \( \bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})), \bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M) \). Then

\[ \phi_1(\alpha, \tilde{\mathcal{M}}) = \sum_{I,J,K} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D, \]

where \([\cdot, \cdot]_D\) is a Dorfman bracket:

\[ [v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v \omega, \]

\[ [\omega, v]_D = -i_v d\omega, \quad [\omega_1, \omega_2]_D = 0. \]

Courant bracket is the antisymmetrized version of \([\cdot, \cdot]_D\).

Similarly:

\[ \phi_2(\alpha, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}) = \tilde{\mathcal{M}} \cdot D\alpha \cdot \tilde{\mathcal{M}} \]

\[ \frac{1}{2} \sum_{I,J,K} \langle b^J, a^K \rangle a^J \otimes \bar{a}^J(\bar{f}^I)\bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J(\bar{f}^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J. \]
One can write formally:

\[ \alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K, \]
\[ \tilde{M} = \sum_I a^I \otimes \bar{a}^I, \]

where \( a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E})) \), \( f^I \in J^\infty(\mathcal{O}_M) \) and \( \bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \), \( \bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M) \). Then

\[ \phi_1(\alpha, \tilde{M}) = \sum_{I,J}[b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D, \]

where \([\cdot, \cdot]_D\) is a Dorfman bracket:

\[ [v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v \omega, \]
\[ [\omega, v]_D = -i_v d\omega, \quad [\omega_1, \omega_2]_D = 0. \]

Courant bracket is the antisymmetrized version of \([\cdot, \cdot]_D\).

Similarly:

\[ \phi_2(\alpha, \tilde{M}, \bar{M}) = \tilde{M} \cdot \Delta \alpha \cdot \bar{M} \]
\[ = \frac{1}{2} \sum_{I,J,K} \langle b^J, \bar{a}^I \rangle a^J \otimes \bar{a}^I (\bar{f}^I) \bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J (f^I) a^K \otimes \langle \bar{b}^J, \bar{a}^I \rangle \bar{a}^J. \]
One can write formally:

$$\alpha = \sum_J f^J \otimes \tilde{b}^J + \sum_K b^K \otimes \tilde{f}^K,$$

$$\tilde{M} = \sum_I a^I \otimes \bar{a}^I,$$

where \(a^I, b^J \in J^\infty(\mathcal{O}(E))\), \(f^I \in J^\infty(\mathcal{O}_M)\) and \(\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{E}))\), \(\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)\). Then

$$\phi_1(\alpha, \tilde{M}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where \([\cdot, \cdot]_D\) is a Dorfman bracket:

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Courant bracket is the antisymmetrized version of \([\cdot, \cdot]_D\).

Similarly:

$$\phi_2(\alpha, \tilde{M}, \tilde{M}) = \tilde{M} \cdot D\alpha \cdot \tilde{M}$$

$$\frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J(\bar{f}^I)\bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J(\bar{f}^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$
Relation to standard second order sigma-model: Let us fill in 0 in $\tilde{\mathcal{M}}$:

$$\mathcal{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}.$$ 

$$S_{fo} = \frac{1}{2\pi i h} \int_\Sigma \left( \langle p \wedge \bar{\partial}X \rangle - \langle \bar{p} \wedge \partial X \rangle - \langle g, p \wedge \bar{p} \rangle - \langle \mu, p \wedge \bar{\partial}X \rangle - \langle \bar{\mu}, \bar{p} \wedge \partial X \rangle - \langle b, \partial X \wedge \bar{\partial}X \rangle \right).$$

Same formulas express symmetries. If $\{g^{ij}\}$ is nondegenerate, then:

$$S_{so} = \frac{1}{4\pi h} \int_\Sigma (G_{\mu\nu}(X) dX^\mu \wedge *dX^\nu + X^* B),$$

$$G_{s\bar{k}} = g_{ij} \bar{\mu}_s^j \mu_{\bar{k}}^i + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{ij} \bar{\mu}_s^j \mu_{\bar{k}}^i - g_{s\bar{k}} - b_{s\bar{k}}$$

$$G_{\bar{s}i} = -g_{ij} \bar{\mu}_s^j - g_{s\bar{i}} \bar{\mu}_s^i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j} \mu_{\bar{i}}^j - g_{\bar{s}i} \mu_{\bar{i}}^j$$

$$B_{\bar{s}i} = g_{\bar{s}j} \mu_\bar{i}^j - g_{ij} \mu_{\bar{i}}^j, \quad B_{\bar{s}\bar{i}} = g_{ij} \mu_{\bar{i}}^j - g_{\bar{s}j} \mu_{\bar{i}}^j.$$

Symmetries $\mathcal{M} \rightarrow \mathcal{M} - D\alpha + \phi_1(\alpha, \mathcal{M}) + \phi_2(\alpha, \mathcal{M}, \mathcal{M})$ are equivalent to:


$$G \rightarrow G - L_\nu G, \quad B \rightarrow B - L_\nu B$$

$$B \rightarrow B - 2d\omega$$

$$\alpha = (\nu, \omega), \quad \nu \in \Gamma(TM), \omega \in \Omega^1(M)$$
Relation to standard second order sigma-model: Let us fill in 0 in \( \tilde{M} \):

\[
\tilde{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}.
\]

\[
S_{so} = \frac{1}{2\pi i h} \int_{\Sigma} \left( \langle p \wedge \bar{\partial}X \rangle - \langle \bar{p} \wedge \partial X \rangle - \langle g, p \wedge \bar{p} \rangle - \langle \mu, p \wedge \bar{\partial}X \rangle - \langle \bar{\mu}, \bar{p} \wedge \partial X \rangle - \langle b, \partial X \wedge \bar{\partial}X \rangle \right).
\]

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\[
S_{so} = \frac{1}{4\pi h} \int_{\Sigma} \left( G_{\mu\nu}(X) dX^{\mu} \wedge *dX^{\nu} + X^* B \right),
\]

\[
G_{\bar{s}k} = g_{ij} \bar{\mu}^i_s \mu^j_k + g_{\bar{s}k} - b_{\bar{s}k}, \quad B_{\bar{s}k} = g_{ij} \bar{\mu}^i_s \mu^j_k - g_{\bar{s}k} - b_{\bar{s}k}
\]

\[
G_{si} = -g_{ij} \bar{\mu}^i_s - g_{sj} \bar{\mu}^j_i, \quad B_{s\bar{i}} = -g_{\bar{s}j} \mu^i_j - g_{\bar{s}i} \mu^i_j
\]

\[
B_{si} = g_{sj} \bar{\mu}^j_i - g_{ij} \bar{\mu}^j_s, \quad B_{\bar{s}i} = g_{\bar{s}j} \mu^j_s - g_{\bar{s}j} \mu^j_i
\]

Symmetries \( \tilde{M} \to \tilde{M} - D\alpha + \phi_1(\alpha, \tilde{M}) + \phi_2(\alpha, \tilde{M}, \tilde{M}) \) are equivalent to:


\[
G \to G - L_v G, \quad B \to B - L_v B
\]

\[
B \to B - 2d\omega
\]

\[
\alpha = (v, \omega), \quad v \in \Gamma(TM), \omega \in \Omega^1(M)
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Same formulas express symmetries. If $\{g^{ij}\}$ is nondegenerate, then:

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$$B_{s\bar{i}} = g_{\bar{s}j} \mu_i^j - g_{\bar{s}j} \bar{\mu}_i^j, \quad B_{\bar{s}\bar{i}} = g_{ij} \mu_i^j - g_{\bar{s}j} \mu_i^j.$$
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$$G_{s\bar{k}} = g_{ij} \bar{\mu}_s \mu^j_k + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{ij} \bar{\mu}_s \mu^j_k - g_{s\bar{k}} - b_{s\bar{k}}$$

$$G_{s\bar{i}} = -g_{ij} \bar{\mu}_s \mu^j_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j} \mu^i_{\bar{\bar{s}}} - g_{\bar{s}j} \mu^i_{\bar{\bar{s}}}$$

$$B_{s\bar{i}} = g_{s\bar{j}} \bar{\mu}_s \mu^i_j, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}j} \mu^i_{\bar{s}} - g_{\bar{s}j} \mu^i_{\bar{s}}.$$

Symmetries $\tilde{M} \to \tilde{M} - D\alpha + \phi_1(\alpha, \tilde{M}) + \phi_2(\alpha, \tilde{M}, \tilde{M})$ are equivalent to:


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Same formulas express symmetries. If $\{g^{ij}\}$ is nondegenerate, then:

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Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on $M$) via sheaves of VOA on $M$ (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set $U$ of $M$ we have VOA:

$$V = \sum_{n=0}^{\infty} V_n, \quad Y : V \to \text{End}(V)[[z, z^{-1}]],$$

generated by:

$$[X^i(z), p_j(w)] = h\delta^i_j \delta(z - w), \quad i, j = 1, 2, \ldots, D/2$$

$$X^i(z) = \sum_{r \in \mathbb{Z}} X^i_r z^{-r}, \quad p_j(z) = \sum_{s \in \mathbb{Z}} p_{j,s} z^{-s-1} \in \text{End}(V)[[z, z^{-1}]],$$

so that

$$V = \text{Span}\{p_{j_1,-s_1}, \ldots, p_{j_k,-s_k} X^i_{-r_1} \ldots X^j_{-r_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$

$$r_m, s_n > 0,$$

$F(U)$ generated by $X^i_0$-modes.
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$$V = \text{Span}\{p_{j_1, -s_1}, \ldots, p_{j_k, -s_k} X^i_{-r_1} \ldots X^i_{-r_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$

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so that

$$V = \text{Span}\{p_{j_1, -s_1}, \ldots, p_{j_k, -s_k} X^i_{-r_1} \ldots X^i_{-r_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$

$$r_m, s_n > 0,$$

$F(U)$ generated by $X^i_0$-modes.
Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $L_0$ is described (under certain constraints on $M$) via sheaves of VOA on $M$ (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set $U$ of $M$ we have VOA:

$$V = \sum_{n=0}^{\infty} V_n, \quad Y : V \rightarrow \text{End}(V)[[z, z^{-1}]],$$

generated by:

$$[X^i(z), p_j(w)] = h\delta^i_j \delta(z - w), \quad i, j = 1, 2, \ldots, D/2$$

$$X^i(z) = \sum_{r \in \mathbb{Z}} X^i_r z^{-r}, \quad p_j(z) = \sum_{s \in \mathbb{Z}} p_{j, s} z^{-s-1} \in \text{End}(V)[[z, z^{-1}]],$$

so that

$$V = \text{Span}\{p_{j_1, -s_1} \ldots, p_{j_k, -s_k} X^i_{-r_1} \ldots X^i_{-r_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$

$$r_m, s_n > 0,$$

$F(U)$ generated by $X^i_0$-modes.
Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on $M$) via sheaves of VOA on $M$ (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set $U$ of $M$ we have VOA:

$$V = \sum_{n=0}^{\infty} V_n, \quad Y : V \to \text{End}(V)[[z, z^{-1}]],$$

generated by:

$$[X^i(z), p_j(w)] = h\delta^i_j \delta(z - w), \quad i, j = 1, 2, \ldots, D/2$$

$$X^i(z) = \sum_{r \in \mathbb{Z}} X^i_r z^{-r}, \quad p_j(z) = \sum_{s \in \mathbb{Z}} p_{j,s} z^{-s-1} \in \text{End}(V)[[z, z^{-1}]],$$

so that

$$V = \text{Span}\{p_{j_1,-s_1}, \ldots, p_{j_k,-s_k} X^i_{-r_1} \ldots X^i_{-r_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$

$$r_m, s_n > 0,$$

$F(U)$ generated by $X^i_0$-modes.
The Virasoro element is:

\[ T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{h} \langle p(z) \partial X(z) \rangle + \partial^2 \phi'(X(z)). \]

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{D}{12} (n^3 - n) \delta_{n,-m} \]

corresponding to correction:

\[ \mathcal{L}_0 \rightarrow \mathcal{L}_{\phi'} = \langle p \wedge \bar{\partial} X \rangle - 2\pi ih R^{(2)}(\gamma) \phi'(X) \]

where \( \phi' = \log \Omega \), where \( \Omega(X) dX^1 \wedge \cdots \wedge dX^n \) is a holomorphic volume form, i.e. for globally defined \( T(z) \), \( M \) has to be Calabi-Yau. The space \( V \) is a lowest weight module for the above Virasoro algebra.

\( V \) can be reproduced from \( V_0 \) and \( V_1 \) as a vertex envelope. The structure of vertex algebra imposes algebraic relations on \( V_0 \oplus V_1 \) giving it a structure of a vertex algebroid.

In our case: \( V_0 \rightarrow \mathcal{O}_U^h = \mathcal{O}_U \otimes \mathbb{C}[h, h^{-1}] \),
\[ V_1 \rightarrow \mathcal{V}^h = \mathcal{V} \otimes \mathbb{C}[h, h^{-1}], \]
\( \mathcal{V} = \mathcal{O}(\mathcal{E}_U) \), generated by : \( v_i(X)p_i \), \( \omega_i(X)\partial X^i \).
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i) $\mathbb{C}$-linear pairing $\mathcal{O}_M \otimes \mathcal{V} \to \mathcal{V}[h]$, i.e. $f \otimes \nu \mapsto f \star \nu$ such that $1 \star \nu = \nu$.

ii) $\mathbb{C}$-linear bracket, satisfying Leibniz algebra $[\ , \ ] : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{V}[h]$,

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iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle \ , \ \rangle : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{O}_M[h]$,

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naturally extending to $\mathcal{O}^h_M$ and $\mathcal{V}^h$, and satisfy the relations

\[
\begin{align*}
  f \star (g \star \nu) - (fg) \star \nu &= \pi(\nu)(f) \star \partial(g) + \pi(\nu)(g) \star \partial(f), \\
  [\nu_1, f \star \nu_2] &= \pi(\nu_1)(f) \star \nu_2 + f \star [\nu_1, \nu_2], \\
  [\nu_1, \nu_2] + [\nu_2, \nu_1] &= \partial\langle \nu_1, \nu_2 \rangle, \quad \pi(f \star \nu) = f \pi(\nu), \\
  \langle f \star \nu_1, \nu_2 \rangle &= f \langle \nu_1, \nu_2 \rangle - \pi(\nu_1)(\pi(\nu_2)(f)), \\
  \pi(\nu)(\langle \nu_1, \nu_2 \rangle) &= \langle [\nu, \nu_1], \nu_2 \rangle + \langle \nu_1, [\nu, \nu_2] \rangle, \\
  \partial(fg) &= f \star \partial(g) + g \star \partial(f), \\
  [\nu, \partial(f)] &= \partial(\pi(\nu)(f)), \quad \langle \nu, \partial(f) \rangle = \pi(\nu)(f),
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\[
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\]

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[v_1, f \ast v_2] = \pi(v_1)(f) \ast v_2 + f \ast [v_1, v_2],
\]

\[
[v_1, v_2] + [v_2, v_1] = \partial\langle v_1, v_2 \rangle, \quad \pi(f \ast v) = f \pi(v),
\]

\[
\langle f \ast v_1, v_2 \rangle = f \langle v_1, v_2 \rangle - \pi(v_1)(\pi(v_2)(f)),
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\[
\pi(v)(\langle v_1, v_2 \rangle) = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle,
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Outline

Sigma-models and conformal invariance conditions

Beltrami-Courant differential

Vertex/Courant algebroids, $G_\infty$-algebras and quasiclassical limit

Einstein Equations
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iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle \, , \rangle : \mathcal{V} \otimes \mathcal{V} \to hO_M[h],$

v) a $\mathbb{C}$-linear map $\partial : O_M \to \mathcal{V}$ such that $\pi \circ \partial = 0,$

naturally extending to $O^h_M$ and $\mathcal{V}^h$, and satisfy the relations

\[
\begin{align*}
\frac{f \ast (g \ast \nu)}{f \ast (g \ast \nu)} - (fg) \ast \nu &= \pi(\nu)(f) \ast \partial(g) + \pi(\nu)(g) \ast \partial(f), \\
[v_1, f \ast \nu_2] &= \pi(v_1)(f) \ast \nu_2 + f \ast [v_1, \nu_2], \\
[v_1, \nu_2] + [\nu_2, v_1] &= \partial\langle v_1, \nu_2 \rangle, \quad \pi(f \ast \nu) = f\pi(\nu), \\
\langle f \ast v_1, \nu_2 \rangle &= f\langle v_1, \nu_2 \rangle - \pi(v_1)(\pi(\nu_2)(f)), \\
\pi(\nu)(\langle v_1, \nu_2 \rangle) &= \langle [v_1, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle, \\
\partial(fg) &= f \ast \partial(g) + g \ast \partial(f), \\
[v, \partial(f)] &= \partial(\pi(\nu)(f)), \quad \langle v, \partial(f) \rangle = \pi(\nu)(f), \\
\end{align*}
\]

where $\nu, v_1, v_2 \in \mathcal{V}^h, f, g \in O^h_M.$
A **vertex** $\mathcal{O}_M$-**algebroid** is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with

i) $\mathbb{C}$-linear pairing $\mathcal{O}_M \otimes \mathcal{V} \rightarrow \mathcal{V}[h]$, i.e. $f \otimes v \mapsto f \ast v$ such that $1 \ast v = v$.

ii) $\mathbb{C}$-linear bracket, satisfying Leibniz algebra $[\ , \ ] : \mathcal{V} \otimes \mathcal{V} \rightarrow h\mathcal{V}[h]$,

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\[
\begin{align*}
f \ast (g \ast v) - (fg) \ast v &= \pi(v)f \ast \partial(g) + \pi(v)g \ast \partial(f), \\
[v_1, f \ast v_2] &= \pi(v_1)f \ast v_2 + f \ast [v_1, v_2], \\
[v_1, v_2] + [v_2, v_1] &= \partial\langle v_1, v_2 \rangle, \quad \pi(f \ast v) = f\pi(v), \\
\langle f \ast v_1, v_2 \rangle &= f\langle v_1, v_2 \rangle - \pi(v_1)(\pi(v_2)(f)), \\
\pi(v)(\langle v_1, v_2 \rangle) &= \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle, \\
\partial(fg) &= f \ast \partial(g) + g \ast \partial(f), \\
[v, \partial(f)] &= \partial(\pi(v)(f)), \quad \langle v, \partial(f) \rangle = \pi(v)(f),
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iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle \ , \ \rangle : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{O}_M[h],$

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naturally extending to $\mathcal{O}^h_M$ and $\mathcal{V}^h$, and satisfy the relations

$$f \ast (g \ast v) - (fg) \ast v = \pi(v)(f) \ast \partial(g) + \pi(v)(g) \ast \partial(f),$$

$$[v_1, f \ast v_2] = \pi(v_1)(f) \ast v_2 + f \ast [v_1, v_2],$$

$$[v_1, v_2] + [v_2, v_1] = \partial\langle v_1, v_2 \rangle, \quad \pi(f \ast v) = f \pi(v),$$

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$$\pi(v)(\langle v_1, v_2 \rangle) = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle,$$

$$\partial(fg) = f \ast \partial(g) + g \ast \partial(f),$$

$$[v, \partial(f)] = \partial(\pi(v)(f)), \quad \langle v, \partial(f) \rangle = \pi(v)(f),$$

where $v, v_1, v_2 \in \mathcal{V}^h, f, g \in \mathcal{O}^h_M.$
A vertex $O_M$-algebroid is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with

i) $\mathbb{C}$-linear pairing $O_M \otimes \mathcal{V} \to \mathcal{V}[h]$, i.e. $f \otimes v \mapsto f \ast v$ such that $1 \ast v = v$.

ii) $\mathbb{C}$-linear bracket, satisfying Leibniz algebra $[ , ] : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{V}[h]$,

iii) $\mathbb{C}$-linear map of Leibniz algebras $\pi : \mathcal{V} \to h\Gamma(TM)[h]$ usually referred to as an anchor

iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle , \rangle : \mathcal{V} \otimes \mathcal{V} \to hO_M[h]$,

v) a $\mathbb{C}$-linear map $\partial : O_M \to \mathcal{V}$ such that $\pi \circ \partial = 0$,

naturally extending to $O_M^h$ and $\mathcal{V}^h$, and satisfy the relations

$$ f \ast (g \ast v) - (fg) \ast v = \pi(v)(f) \ast \partial(g) + \pi(v)(g) \ast \partial(f), $$

$$ [v_1, f \ast v_2] = \pi(v_1)(f) \ast v_2 + f \ast [v_1, v_2], $$

$$ [v_1, v_2] + [v_2, v_1] = \partial\langle v_1, v_2 \rangle, \quad \pi(f \ast v) = f\pi(v), $$

$$ \langle f \ast v_1, v_2 \rangle = f\langle v_1, v_2 \rangle - \pi(v_1)(\pi(v_2)(f)), $$

$$ \pi(v)\langle v_1, v_2 \rangle = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle, $$

$$ \partial(fg) = f \ast \partial(g) + g \ast \partial(f), $$

$$ [v, \partial(f)] = \partial(\pi(v)(f)), \quad \langle v, \partial(f) \rangle = \pi(v)(f), $$

where $v, v_1, v_2 \in \mathcal{V}^h$, $f, g \in O_M^h$. 
A vertex $\mathcal{O}_M$-algebroid is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with

i) $\mathbb{C}$-linear pairing $\mathcal{O}_M \otimes \mathcal{V} \to \mathcal{V}[h]$, i.e. $f \otimes v \mapsto f \ast v$ such that $1 \ast v = v$.

ii) $\mathbb{C}$-linear bracket, satisfying Leibniz algebra $[\ , \ ] : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{V}[h]$,

iii) $\mathbb{C}$-linear map of Leibniz algebras $\pi : \mathcal{V} \to h\Gamma(TM)[h]$ usually referred to as an anchor

iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle \ , \ \rangle : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{O}_M[h]$,

v) a $\mathbb{C}$-linear map $\partial : \mathcal{O}_M \to \mathcal{V}$ such that $\pi \circ \partial = 0$,

naturally extending to $\mathcal{O}_M^h$ and $\mathcal{V}^h$, and satisfy the relations

\[
\begin{align*}
  f \ast (g \ast v) - (fg) \ast v &= \pi(v)(f) \ast \partial(g) + \pi(v)(g) \ast \partial(f), \\
  [v_1, f \ast v_2] &= \pi(v_1)(f) \ast v_2 + f \ast [v_1, v_2], \\
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  \pi(v)(\langle v_1, v_2 \rangle) &= \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle, \\
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iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle \ , \ \rangle : \mathcal{V} \otimes \mathcal{V} \rightarrow h\mathcal{O}_M[h]$,

v) a $\mathbb{C}$-linear map $\partial : \mathcal{O}_M \rightarrow \mathcal{V}$ such that $\pi \circ \partial = 0$,

naturally extending to $\mathcal{O}_M^h$ and $\mathcal{V}^h$, and satisfy the relations

\[
f \ast (g \ast v) - (fg) \ast v = \pi(v)(f) \ast \partial(g) + \pi(v)(g) \ast \partial(f),
\]

\[
[v_1, f \ast v_2] = \pi(v_1)(f) \ast v_2 + f \ast [v_1, v_2],
\]

\[
[v_1, v_2] + [v_2, v_1] = \partial\langle v_1, v_2 \rangle, \quad \pi(f \ast v) = f\pi(v),
\]

\[
\langle f \ast v_1, v_2 \rangle = f\langle v_1, v_2 \rangle - \pi(v_1)(\pi(v_2)(f)),
\]

\[
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\[
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\]

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where $v, v_1, v_2 \in \mathcal{V}^h$, $f, g \in \mathcal{O}_M^h$. 
For our considerations $\mathcal{V} = \mathcal{O}(\xi)$:

$$
\partial f = df, \quad \pi(v)f = -hv(f), \quad \pi(\omega) = 0,
$$

$$
f * v = fv + hdX^i \partial_i \partial_j fv^j, \quad f * \omega = f \omega,
$$

$$
[v_1, v_2] = -h[v_1, v_2]_D - h^2 dX^i \partial_i \partial_k v_1^s \partial_s v_2^k,
$$

$$
[v, \omega] = -h[v, \omega]_D, \quad [\omega, v] = -h[\omega, v]_D, \quad [\omega_1, \omega_2] = 0,
$$

$$
\langle v, \omega \rangle = -h\langle v, \omega \rangle^s, \quad \langle v_1, v_2 \rangle = -h^2 \partial_i v_1^i \partial_j v_2^j, \quad \langle \omega_1, \omega_2 \rangle = 0,
$$

where $v$ and $\omega$ are vector fields and 1-forms correspondingly.

Together with $\text{div}_{\phi'}$—the divergence operator with respect to $\phi'$—these operations generate vertex algebroid with Calabi-Yau structure.
Einstein field equations, Courant algebroids and Homotopy algebras

Anton Zeitlin

Outline
Sigma-models and conformal invariance conditions
Beltrami-Courant differential
Vertex/Courant algebroids, $G_\infty$-algebras and quasiclassical limit
Einstein Equations

For our considerations $\mathcal{V} = \mathcal{O}(\mathcal{E})$:

\[
\partial f = df, \quad \pi(v)f = -hv(f), \quad \pi(\omega) = 0, \\
f \ast v = fv + hdX^i \partial_i \partial_j f v^j, \quad f \ast \omega = f \omega, \\
[v_1, v_2] = -h[v_1, v_2]_D - h^2 dX^i \partial_i \partial_k v_1^s \partial_s v_2^k, \\
[v, \omega] = -h[v, \omega]_D, \quad [\omega, v] = -h[\omega, v]_D, \quad [\omega_1, \omega_2] = 0, \\
\langle v, \omega \rangle = -h\langle v, \omega \rangle^s, \quad \langle v_1, v_2 \rangle = -h^2 \partial_i v_1^i \partial_j v_2^j, \quad \langle \omega_1, \omega_2 \rangle = 0,
\]

where $v$ and $\omega$ are vector fields and 1-forms correspondingly.

Together with $\text{div}_{\phi'}$—the divergence operator with respect to $\phi'$ these operations generate vertex algebroid with Calabi-Yau structure.
Vertex algebra $V$ is a Virasoro module. The corresponding semi-infinite complex $V^{semi}$ (the analogue of Chevalley complex for Virasoro algebra) is a vertex algebra too:

$$V^{semi} = V \otimes \Lambda,$$

$\Lambda$ generated by $[b(z), c(w)]_+ = \delta(z - w)$.

The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1} = c(z) T(z) + : c(z) \partial c(z) b(z) :$$

is nilpotent when $D = 26$ (famous dimension 26!). However, we will consider subcomplex of light modes (i.e. $L_0 = 0$) denoted in the following as $(\mathcal{F}_h^\ast, Q)$, where we can drop this condition:
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is nilpotent when $D = 26$ (famous dimension 26!). However, we will consider subcomplex of light modes (i.e. $L_0 = 0$) denoted in the following as $(\mathcal{F}'_h, Q)$, where we can drop this condition:
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The corresponding differential

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The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

\[(A, B)_h = \text{Res}_z \frac{A(z)B}{z}\]

\[Q(a_1, a_2)_h = (Qa_1, a_2)_h + (-1)^{|a_1|}(a_1, Qa_2)_h,\]

\[(a_1, a_2)_h - (-1)^{|a_1||a_2|}(a_2, a_1)_h =\]

\[Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2),\]

\[Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|}(a_1, Qa_2, a_3)_h +\]

\[(-1)^{|a_1|+|a_2|}(a_1, a_2, Qa_3)_h = ((a_1, a_2)_h, a_3)_h - (a_1, (a_2, a_3)_h)_h\]

Operator \(b\) of degree -1 (0-mode of \(b(z)\)) on \((\mathcal{F}_h, Q)\) which anticommutes with \(Q\):

\[\mathcal{V}^h \xleftarrow{-id} \mathcal{V}^h\]

\[\mathcal{O}^h_M \xleftarrow{id} \mathcal{O}^h_M\]

\[\mathcal{O}^h_M \xrightarrow{-id} \mathcal{O}^h_M\]
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\[Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|} m(a_1, Qa_2),\]
\[Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|} (a_1, Qa_2, a_3)_h +\]
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Operator \(b\) of degree -1 (0-mode of \(b(z)\)) on \((\mathcal{F}_h, Q)\) which anticommutes with \(Q\):

\[\mathcal{V}^h \leftarrow -\text{id} \mathcal{V}^h\]
\[\bigoplus \mathcal{O}^h_M \leftarrow \bigoplus \mathcal{O}^h_M\]
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\[(a_1, a_2)_h - (-1)^{|a_1||a_2|}(a_2, a_1)_h = Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2),\]

\[Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|}(a_1, Qa_2, a_3)_h +
(-1)^{|a_1|+|a_2|}(a_1, a_2, Qa_3)_h = ((a_1, a_2)_h, a_3)_h - (a_1, (a_2, a_3)_h)_h\]

Operator \(b\) of degree -1 (0-mode of \(b(z)\)) on \((\mathcal{F}_h, Q)\) which anticommutes with \(Q\):

\[\mathcal{V}_h \overset{-\text{id}}{\leftrightarrow} \mathcal{V}_h\]

\[\mathcal{O}_M^h \overset{\text{id}}{\leftrightarrow} \mathcal{O}_M^h\]
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The homotopy associative and homotopy commutative product of Lian and Zuckerman:

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\[(a_1, a_2)_h - (-1)^{|a_1||a_2|}(a_2, a_1)_h =\]

\[Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2),\]

\[Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|}(a_1, Qa_2, a_3)_h +\]

\[(-1)^{|a_1|+|a_2|}(a_1, a_2, Qa_3)_h = ((a_1, a_2)_h, a_3)_h - (a_1, (a_2, a_3)_h)_h\]

Operator \(b\) of degree -1 \((0\text{-mode of } b(z))\) on \((\mathcal{F}_h, Q)\) which anticommutes with \(Q\):

\[\forall h \xleftarrow{-\text{id}} \forall h\]

\[\bigoplus \quad \bigoplus\]

\[\mathcal{O}_M^h \xleftarrow{\text{id}} \mathcal{O}_M^h \quad \mathcal{O}_M^h \xleftarrow{-\text{id}} \mathcal{O}_M^h\]
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\[ Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|} m(a_1, Qa_2), \]
\[ Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|} (a_1, Qa_2, a_3)_h + \]
\[ (-1)^{|a_1|+|a_2|} (a_1, a_2, Qa_3)_h = ((a_1, a_2)_h, a_3)_h - (a_1, (a_2, a_3)_h)_h \]

Operator \( b \) of degree -1 (0-mode of \( b(z) \)) on \((\mathcal{F}_h, Q)\) which anticommutes with \( Q \):

\[ \mathcal{V}^h \leftarrow -id \mathcal{V}^h \]
\[ \bigoplus \bigoplus \]
\[ \mathcal{O}^h_M \leftarrow id \mathcal{O}^h_M \quad \mathcal{O}^h_M \leftarrow -id \mathcal{O}^h_M \]
One can define a bracket:

\[ (-1)^{|a_1|}\{a_1, a_2\}_h = b(a_1, a_2)_h - (b a_1, a_2)_h - (-1)^{|a_1|}(a_1 b a_2)_h, \]

so that together with \( Q, (\cdot, \cdot)_h \) it satisfies the relations of homotopy Gerstenhaber algebra:

\[
\begin{align*}
\{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\}_h &= \{a_1, a_2\}_h + (-1)^{|a_1|-1}|a_2|\{a_2, a_1\}_h, \\
\{(a_1, a_2)_h, a_3\}_h &= \{(a_1, a_2)_h, a_3\}_h + (-1)^{|a_1|-1}a_1 \{a_1, a_3\}_h, \\
\{(a_1, a_2)_h, a_3\}_h - (a_1, \{a_2, a_3\}_h)_h &= (-1)^{|a_3|-1}a_2 \{a_1, a_3\}_h, \\
\{\{a_1, a_2\}_h, a_3\}_h - \{a_1, \{a_2, a_3\}_h\}_h &= (-1)^{|a_1|+|a_2|-1}Qn_h(a_1, a_2, a_3) - n_h(Qa_1, a_2, a_3) - (-1)^{|a_1|+|a_2|}n_h(a_1, Qa_2, a_3), \\
\{a_1, a_2\}_h, a_3\}_h - \{a_1, \{a_2, a_3\}_h\}_h &= (-1)^{|a_1|-1}(a_1 b a_2)_h - (-1)^{|a_1|-1}(b a_1, a_2)_h - (-1)^{|a_1|-1}(a_1 b a_2)_h = 0.
\end{align*}
\]

The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to \( G_\infty \)-algebra.
One can define a bracket:

\[ (-1)^{|a_1|} \{a_1, a_2\}_h = b(a_1, a_2)_h - (b a_1, a_2)_h - (-1)^{|a_1|} (a_1 b a_2)_h, \]

so that together with \( Q, (\cdot, \cdot)_h \) it satisfies the relations of homotopy Gerstenhaber algebra:

\[
\begin{align*}
\{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)} \{a_2, a_1\}_h &= \\
(-1)^{|a_1|-1} (Q m'_h(a_1, a_2) - m'_h(Q a_1, a_2) - (-1)^{|a_2|} m'_h(a_1, Q a_2)), \\
\{a_1, (a_2, a_3)_h\}_h &= (\{a_1, a_2\}_h, a_3)_h + (-1)^{(|a_1|-1)|a_2|} (a_2, \{a_1, a_3\}_h)_h, \\
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(-1)^{|a_1|+|a_2|-1} (Q n'_h(a_1, a_2, a_3) - n'_h(Q a_1, a_2, a_3) - \\
(-1)^{|a_1|} n'_h(a_1, Q a_2, a_3) - (-1)^{|a_1|+|a_2|} n'_h(a_1, a_2, Q a_3), \\
\{\{a_1, a_2\}_h, a_3\}_h - \{a_1, \{a_2, a_3\}_h\}_h &= \\
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{a_1, (a_2, a_3)_h} &= ((a_1, a_2)_h, a_3)_h + (-1)^{|a_1|-1)(|a_2|) (a_2, \{a_1, a_3\}_h)_h, \\
\{(a_1, a_2)_h, a_3\}_h - (a_1, \{a_2, a_3\}_h)_h - (-1)^{|a_3|-1)(|a_2|) \{a_1, a_3\}_h, a_2\}_h &= (-1)^{|a_1|+|a_2|-1}(Qn'_h(a_1, a_2, a_3) - n'_h(Qa_1, a_2, a_3) - \]

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Homotopy algebras: $G_\infty$, $L_\infty$, $C_\infty$

Let $A$ be a graded vector space, consider free graded Lie algebra $\text{Lie}(A)$.

\[ \text{Lie}^{k+1}(A) = [A, \text{Lie}^k A], \quad \text{Lie}^1(A) = A. \]

Consider free graded commutative algebra $GA$ on the suspension $(\text{Lie}(A))[-1]$, i.e.

\[ GA = \bigoplus_n \bigwedge^n \text{Lie}(A)[-n] \]

There are natural $[\cdot, \cdot], \wedge$ operations on $GA$ of degree -1, 0 correspondingly, generating a Gerstenhaber algebra.

A $G_\infty$-algebra (Tamarkin, Tsygan, 2000) is a graded space $V$ with a differential $\partial$ of degree 1 of $G(V[1]^*)$, such that $\partial$ is a derivation w.r.t bracket and the product.

Multiplicative Ideals, preserved by $\partial$: $l_1$-generated by the commutant of $\text{Lie}(V[1]^*)$, $l_2 = \bigwedge_{n \geq 2} (\text{Lie}(V[1]^*))[-n]$. That induces differentials on corresponding factors: $\bigwedge_{n \geq 1} (V[1]^*)[-n]$ and $\text{Lie}(V[1]^*)[-1]$. The resulting structures on $V$ are called $L_\infty$-algebra and $C_\infty$-algebra correspondingly.
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Restriction of $\partial$ on $V[1]^*$:

$$V[1]^* \rightarrow \text{Lie}^{k_1}(V[1]^*) \wedge \cdots \wedge \text{Lie}^{k_n}(V[1]^*).$$

Conjugated map:

$$m_{k_1,k_2,...,k_n} : V^{\otimes k_1} \otimes \cdots \otimes V^{\otimes k_n} \rightarrow V.$$

of degree $3 - n - k_1 - \cdots - k_n$, satisfying bilinear relations.

In our previous notation $m_1 = Q$, $m_2$-symmetrized LZ product, $m_1,1$–antisymmetrized LZ bracket.

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An important feature of $L_\infty$ algebra is a Maurer-Cartan equation ($\Phi$ is of degree 2):

$$Q \Phi + \sum_{n \geq 2} \frac{1}{n!} [\Phi, \ldots, \Phi] + \cdots = 0,$$

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Quasiclassical limit of LZ $G_\infty$ algebra

The following complex $(\mathcal{F}^\cdot, Q)$:

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$$ (\cdot, \cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot, \cdot\}_h : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}^{i+j-1}[h], $$

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so that

$$ (\cdot, \cdot)_0 = \lim_{h \to 0} (\cdot, \cdot)_h, \quad \{\cdot, \cdot\}_0 = \lim_{h \to 0} h^{-1}\{\cdot, \cdot\}_h, \quad b_0 = \lim_{h \to 0} h^{-1}b $$

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\[
\begin{array}{ccc}
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\oplus & \xrightarrow{\oplus} & \oplus \\
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is a subcomplex of $(\mathcal{F}_h, Q)$. Then

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\end{align*}
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The resulting $C_\infty$ and $L_\infty$ algebras are reduced to $C_3$ and $L_3$ algebras.

Conjecture: This $G_\infty$-algebra is the $G_3$-algebra (no homotopies beyond trilinear operations).

Classical limit procedure for vertex algebroid (due to P. Bressler):

\[ [\cdot, \cdot]_0 = \lim_{h \to 0} \frac{1}{h} [\cdot, \cdot], \quad \pi_0 = \lim_{h \to 0} \frac{1}{h} \pi, \quad \langle \cdot, \cdot \rangle_0 = \lim_{h \to 0} \frac{1}{h} \langle \cdot, \cdot \rangle. \]

The resulting operations form a Courant algebroid (Z.-J. Liu, A. Weinstein, P. Xu, 1997)
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$[\cdot, \cdot]_0 = \lim_{h \to 0} \frac{1}{h} [\cdot, \cdot], \pi_0 = \lim_{h \to 0} \frac{1}{h} \pi, \langle \cdot, \cdot \rangle_0 = \lim_{h \to 0} \frac{1}{h} \langle \cdot, \cdot \rangle.$

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The symmetrized operations $(\cdot, \cdot)_0, \{\cdot, \cdot\}_0, \ldots$ satisfy the relations of the homotopy Gerstenhaber algebra, so that all non-covariant higher-order terms disappear from the multilinear operations.

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A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with a structure of a Leibniz $\mathbb{C}$-algebra $[\cdot, \cdot]_0 : \mathcal{Q} \otimes \mathcal{Q} \to \mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0 : \mathcal{Q} \to \Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle \cdot, \cdot \rangle : \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{Q} \to \mathcal{O}_M$, a derivation $\partial : \mathcal{O}_M \to \mathcal{Q}$ which satisfy

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for $f \in \mathcal{O}_M$ and $q, q_1, q_2 \in \mathcal{Q}$.

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In our case $\mathcal{Q} \cong \mathcal{O}(E)$, $\pi_0$ is just a projection on $\mathcal{O}(TM)$

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We show that it is a part of a more general structure, homotopy Gerstenhaber algebra.

**Question:** Is there a direct path (avoiding vertex algebra) from Courant algebroid to $G_3$-algebra? Odd analogue of Manin double?

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Simplest version: $G_\infty \to$ Gerstenhaber algebra

Subcomplex $(\mathcal{F}_sm, Q)$:

\[
\begin{align*}
\mathcal{O}(T^{(1,0)}M) &\quad \mathcal{O}(T^{(1,0)}M) \\
\mathcal{O}_M &\quad \mathcal{O}_M
\end{align*}
\]

\[
\begin{align*}
\oplus &\quad \oplus \\
\downarrow 0 &\quad \downarrow \frac{1}{2} \text{div} \\
\mathbb{C} &\quad \mathbb{C}
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\]

The $G_\infty$ algebra degenerates to $G$-algebra. Moreover, due to $b_0$ it is a BV-algebra. Combine chiral and antichiral part:

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Components: $(g, \bar{\nu}, \nu, \phi, \bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

$$\Gamma(T^{(1,0)}(M)), \Gamma(T^{(0,1)}(M)) \text{ components are correspondingly holomorphic and antiholomorphic}.$$

1). Vector field $div_\Omega g$, where $\log \Omega = -2\Phi_0 = -2(\phi' + \bar{\phi}' + \phi + \bar{\phi})$ and $\partial_i \partial_j \Phi_0 = 0$, is such that its $\Gamma(T^{(1,0)}(M)), \Gamma(T^{(0,1)}(M))$ components are correspondingly holomorphic and antiholomorphic.

2). Bivector field $g \in \Gamma(T^{(1,0)}(M) \otimes T^{(0,1)}(M))$ obeys the following equation:

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where $\mathcal{L}_{\text{div}_\Omega}(g)$ is a Lie derivative with respect to the corresponding vector fields and

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Physically:

\[
\int [dp][d\bar{p}][dX][d\bar{X}] e^{-\frac{1}{2\pi i\hbar} \int_{\Sigma} (\langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial X \rangle - \langle g, p \wedge \bar{p} \rangle) + \int_{\Sigma} R^{(2)}(\gamma) \Phi_0(X)} =
\]

\[
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Main Conjecture

Consider

$$ \mathbf{F}_b^- = \mathcal{F} \otimes \mathcal{F} |_{b^- = 0} $$

with the $L_\infty$-algebra structure given by Lian-Zuckerman construction. One can explicitly check that GMC symmetry

$$ (\Psi = \Psi(M, \Phi, \text{auxiliary fields}) $$

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Conjecture: The corresponding Maurer-Cartan equation gives Einstein equations on $G, B, \Phi$ expressed in terms of Beltrami-Courant differential. The symmetries of the Maurer-Cartan equation reproduce mentioned above symmetries of Einstein equations.
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Thank you!