Quantum Integrable Systems via Quantum K-theory

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Introduction

We will talk about the relationship between two seemingly independent areas of mathematics:

- Quantum Integrable Systems

  Exactly solvable models of statistical physics: spin chains, vertex models

  1930s: Hans Bethe: Bethe ansatz solution of Heisenberg model

  1960-70s: R.J. Baxter, C.N. Young: Yang-Baxter equation, Baxter operator

  1980s: Development of "QISM" by Leningrad school leading to the discovery of quantum groups by Drinfeld and Jimbo

  Since 1990s: textbook subject and an established area of mathematics and physics.

- Enumerative geometry: quantum K-theory

  Generalization of quantum cohomology in the early 2000s by A. Givental, Y.P. Lee and collaborators. Recently big progress in this direction by A. Okounkov and his school.

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Path to this relationship:

- First hints: work of Nekrasov and Shatashvili on 3-dimensional gauge theories, now known as Gauge-Bethe correspondence:

  N. Nekrasov, S. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, arXiv:0901.4744
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- Subsequent work in geometric representation theory:

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Understanding (enumerative) geometry of symplectic resolutions:

"Lie algebras of XXI century" (A. Okounkov' 2012)

Important examples: Springer resolution, Hilbert scheme of points in the plane, Hypertoric varieties,...

A large class of symplectic resolutions is provided by Nakajima quiver varieties (simplest subclass: $T^*Gr(k,n)$)

In this talk our main example will be $T^*Gr(k,n)$ and more generally, cotangent bundles to (partial) flag varieties.

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Based on:

- Petr P. Pushkar, Andrey Smirnov, A.Z., *Baxter Q-operator from quantum K-theory*, arXiv:1612.08723
- Peter Koroteev, Petr P. Pushkar, Andrey Smirnov, A.Z., *Quantum K-theory of Quiver Varieties and Many-Body Systems*, arXiv:1705.10419
- Peter Koroteev, Anton M. Zeitlin, Difference Equations for K-theoretic Vertex Functions of Type-A Nakajima Varieties arXiv:1802.04463

and to some extent on

- Peter Koroteev, Daniel S. Sage, Anton M. Zeitlin, (SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality arXiv:1811.09937

Outline

Loop algebras and evaluation modules

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules* form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
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Quantum groups

Quantum group

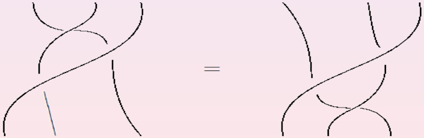
$$U_\hbar(\hat{\mathfrak{g}})$$

is a deformation of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1) \otimes V_2(a_2)$$

$$V_2(a_2) \otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:



The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices (the so-called FRT construction).

Integrability and Baxter algebra

Source of integrability: commuting *transfer matrices*, generating *Baxter algebra* which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

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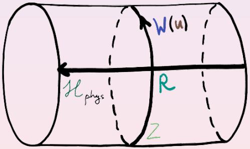
over auxiliary $W(u)$ space:

$$T_W(u) = \mathrm{Tr}_{W(u)}\Big((Z \otimes 1)\ \tilde{R}_{W(u),\mathcal{H}_{phys}}\Big)$$



Here $Z \in e^{\mathfrak{h}}$, where $\mathfrak{h} \in \mathfrak{g}$ are diagonal matrices.

Integrability:

$$[T_{W'}(u'), T_W(u)] = 0$$

There are special transfer matrices is called *Baxter Q-operators*. Such operators generate all Baxter algebra.

Primary goal for physicists is to diagonalize $\{T_W(u)\}$ simultaneously.

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$\mathfrak{g} = \mathfrak{sl}(2)$: XXZ spin chain

Textbook example (and main example in this talk) is XXZ Heisenberg spin chain:

$$\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

States:

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Here $\mathbb{C}^2$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic method to diagonalize transfer matrices:

Algebraic Bethe ansatz

as a part of Quantum Inverse Scattering Method developed in the 1980s.

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Bethe equations and Q-operator

The eigenvalues are generated by symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}\,, \quad i = 1\cdots k,$$

so that the eigenvalues $\Lambda(u)$ of the Q-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\Lambda(u) = \prod_{i=1}^{k} (1 + u \cdot x_i)$$

A real challenge is to describe representation-theoretic meaning of Q-operator for general $\mathfrak{g}$ (possibly infinite-dimensional).

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Modern way of looking at Bethe ansatz: solving q-difference equations for

$$\Psi(z_1,\dots,z_k;a_1,\dots,a_n)\in V_1(a_1)\otimes\dots\otimes V_n(a_n)[[z_1,\dots,z_k]]$$

known as

Quantum Knizhnik-Zamolodchikov (aka Frenkel-Reshetikhin) equations:

$$\Psi(qa_1,\dots,a_n,\{z_i\})=(Z\otimes 1\otimes\dots\otimes 1)R_{V_1,V_n}\dots R_{V_1,V_2}\Psi$$
$$+$$
$$\text{commuting difference equations in } z-\text{variables}$$

Here $\{z_i\}$ are the components of twist variable Z.

The latter series of equations are known as dynamical equations, studied by Etingof, Felder, Tarasov, Varchenko, ...

In $q\to 1$ limit we arrive to an eigenvalue problem. Studying the asymptotics of the corresponding solutions we arrive to Bethe equations and eigenvectors.

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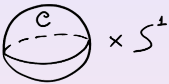
Nekrasov-Shatashvili ideas

In 2009 Nekrasov and Shatashvili looked at 3d SUSY gauge theories on $\mathcal{C} \times S^1$:



with gauge group

$$G = U(v_1) \times U(v_2) \times \dots U(v_{rank\,\mathfrak{g}}),$$

and some "matter fields" (sections of associated vector G-bundles), to be specified below.

The collection $\{v_i\}$ determines the weights of the corresponding subspace in $\mathcal{H}$.

In the simplest case of $\mathfrak{g} = \mathfrak{sl}(2)$ we just have one $U(v)$ and

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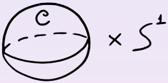

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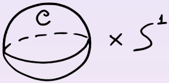

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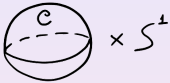

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Full Gauge/Bethe correspondence dictionary

Gauge group G: $U(v_1) \times U(v_2) \times \dots U(v_{rank\,\mathfrak{g}})$

The set $\{v_i\}$ determines the weight (e.g. number of inverted spins)

Maximal torus: $\{x_{i_1}, \dots, x_{i_{v_i}}\}$ — these are Bethe roots variables.

Matter Fields: affine space $\mathcal{M}$

- Standard matter fields: $\oplus_{i=1}^{rank\,\mathfrak{g}} V_i^* \otimes W_i$, s.t. $dim(V_i) = v_i$;

  W_i is a *framing ("flavor")* space, where $\mathbb{C}^\times_{a_1} \times \mathbb{C}^\times_{a_2} \times \dots$ act.

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Maximal torus: $\{x_{i_1},\dots,x_{i_{v_i}}\}$ — these are Bethe roots variables.

Matter Fields: affine space $\mathcal{M}$

- Standard matter fields: $\oplus_{i=1}^{rank\,\mathfrak{g}} V_i^*\otimes W_i$, s.t. $dim(V_i)=v_i$;

  W_i is a *framing ("flavor")* space, where $\mathbb{C}^\times_{a_1}\times\mathbb{C}^\times_{a_2}\times\dots$ act.

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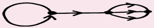

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Full Gauge/Bethe correspondence dictionary

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  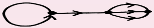

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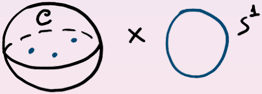

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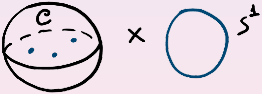

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$$\mathrm{str}(e^{-\beta \not{D}^2} A) = \mathrm{tr}_{\mathrm{Ker} \not{D}_{even}}(A) - \mathrm{tr}_{\mathrm{Ker} \not{D}_{odd}}(A) = \mathrm{str}_{\mathrm{index} \not{D}}(A)$$

Mathematically those correspond to (very similar to GW curve counting!) weighted K-theoretic counts of quasimaps:

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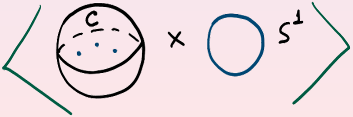
One can think of quantum K-theory ring:



Key Ideas

Nekrasov and Shatashvili:

$$\mathrm{Quantum\ K-theory\ ring\ of\ Nakajima\ variety} =$$

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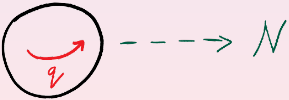
Input by Okounkov:

$$q - \text{difference equations } = \text{qKZ equations} + \text{ dynamical equations}$$



In the following we will talk about this in the simplest case:

- Nakajima variety: $N = T^*Gr(k,n)$
- Quantum Integrable System: $\mathfrak{sl}(2)$ XXZ spin chain.

Notation

$$T^*Gr(k,n) = N_{k,n}, \quad \sqcup_k N_{k,n} = N(n).$$

As a Nakajima variety:

$$N_{k,n} = T^*\mathcal{M} /\!\!/\!\!/\!\!/ GL(V) = \mu^{-1}(0)_s / GL(V),$$

where

$$T^*\mathcal{M} = Hom(V,W) \oplus Hom(W,V)$$

Tautological bundles:

$$\mathcal{V} = T^*\mathcal{M} \times V /\!\!/\!\!/\!\!/ GL(V), \quad \mathcal{W} = T^*\mathcal{M} \times W /\!\!/\!\!/\!\!/ GL(V)$$

For any $\tau \in K_{GL(V)}(\cdot) = \Lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_k^{\pm 1})$ we introduce a tautological bundle:

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Tori, Fixed points and Bethe roots

Torus action:

$$A = \mathbb{C}^{\times}_{a_1} \times \cdots \times \mathbb{C}^{\times}_{a_n} \circlearrowright W,$$

$$\text{Full torus}: \quad T = A \times \mathbb{C}^{\times}_{\hbar}, \text{ where } \mathbb{C}^{\times}_{\hbar} \text{ scales cotangent directions}$$

Fixed points: $\mathbf{p} = \{s_1, \dots, s_k\} \in \{a_1, \dots, a_n\}$

Denote $\mathcal{A} := \mathbb{Q}(a_1, \dots, a_n, \hbar)$, $R := \mathbb{Z}(a_1, \dots, a_n, \hbar)$, then localized K-theory is:

$$K_T(N(n))_{loc} = K_T(N(n)) \otimes_R \mathcal{A} = \sum_{k=0}^{n} K_T(N_{k,n}) \otimes_R \mathcal{A}$$

is a 2^n-dimensional $\mathcal{A}$-vector space (Hilbert space for spin chain), spanned by $\mathcal{O}_{\mathbf{p}}$.

Classical Bethe equations: The eigenvalues of the operators of multiplication by τ are $\tau(x_1, \cdots, x_k)$ evaluated at the solutions of the following equations:

$$\prod_{j=1}^{n} (x_i - a_j) = 0, \quad i = 1, \dots, k, \ with \ x_i \neq x_j$$

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Quantum tautological classes and Bethe equations

We will use theory of quasimaps:

$$\mathcal{C} - -- \to N_{k,n}$$

in order to deform tensor product: $A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B \, z^d$.

We will also define quantum tautological classes:

$$\hat{\tau}(z) = \tau + \sum_{d=1}^{\infty} \tau_d z^d \in K_T(N(n))[[z]]$$

Theorem. [P. Pushkar, A. Smirnov, A.Z] The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1, \dots, x_k)$ evaluated at the solutions of the following equations:

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The quantum K-theoretic meaning of the Q-operator

Theorem. [P. Pushkar, A. Smirnov, A.Z.]

- The quantum multiplication on quantum tautological class corresponding to $\tau_u := \oplus_{m\geq 0} u^m \Lambda^m \mathcal{V}$ coincides with Q-operator, i..e

$$\hat{\tau}_u(z) = Q(u)$$

- Explicit universal formulas for quantum products::

$$\widehat{\Lambda^\ell \mathcal{V}}(z) = \Lambda^\ell \mathcal{V} + a_1(z)\ F_0 \Lambda^{\ell-1}\mathcal{V} E_{-1} + \dots + a_\ell(z)\ F_0^\ell E_{-1}^\ell,$$

where $a_m(z) = \frac{(\hbar-1)^m\ \hbar^{\frac{m(m+1)}{2}} K^m}{(m)_\hbar! \prod_{i=1}^{m} (1-(-1)^n z^{-1} \hbar^i K)}$,

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- vector bundle $\mathscr{V}$ on $\mathcal{C}$ of rank k.
- section $f \in H^0(C, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$, satisfying the condition $\mu = 0$, where $\mathscr{M} = Hom(\mathscr{V}, \mathscr{W})$, so that $\mathscr{W}$ is a trivial bundle of rank n.

$$ev_p(f) = f(p) \in [\mu^{-1}(0)/GL(V)] \supset N_{k,n}$$

Quasimap is *stable* if $f(p) \in N_{k,n}$ for all but finitely many points, known as *singularities* of quasimap.

For the moduli space of quasimaps

$$QM(N_{k,n}) = \text{stable quasimaps to } N_{k,n}/\sim$$

only $\mathscr{V}$ and f vary, while $\mathcal{C}$ and $\mathscr{W}$ remain the same.

$$deg(f) := deg(\mathscr{V}), \quad QM(N_{k,n}) = \sqcup_{d\geq 0} QM^d(N_{k,n}).$$

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$$QM(N_{k,n}) = \text{stable quasimaps to } N_{k,n}/\sim$$

only $\mathscr{V}$ and f vary, while $\mathcal{C}$ and $\mathscr{W}$ remain the same.

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Relative quasimaps

Resolution, to make evaluation map proper:

$$QM^d(N_{k,n})_{\text{nonsing } p} \hookrightarrow QM^d(N_{k,n})_{\text{relative } p} \xrightarrow{\widetilde{\text{ev}}_p} N_{k,n}$$

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$$
\begin{array}{ccc}
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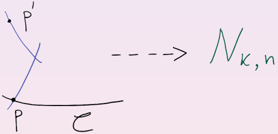
That allows the curve to break: emergence of "*accordeons*":



$$\begin{array}{ccccc} p' & \longrightarrow & \mathcal{C}' & \overset{f'}{\dashrightarrow} & N_{k,n} \\ \downarrow & & \downarrow \pi & & \\ p & \longrightarrow & \mathcal{C} & & \end{array}$$

i)π is a stabilization of $(\mathcal{C}', \mathrm{p}')$

ii)f' : nonsing at p' and nodes of $\mathcal{C}'$

iii)$\mathrm{Aut}(\mathrm{f}')$ is finite

Virtual sheaves

$QM^d(N_{k,n})$ have perfect deformation-obstruction theory:

- If $(\mathscr{V}, \mathscr{W})$ defines quasimap nonsingular at p,

$$T^{\rm vir}_{(\mathscr{V},\mathscr{W})} {QM^d}_{\text{nonsing p}}(N_{k,n}) = \mathrm{Def} - \mathrm{Obs} = H^{\bullet}(\mathscr{P} \oplus \hbar\, \mathscr{P}^*),$$

  where $\mathscr{P}$ is the polarization bundle on the curve $\mathcal{C}$:

$$\mathscr{P} = \mathscr{W} \otimes \mathscr{V}^* - \mathscr{V}^* \otimes \mathscr{V}.$$

- Virtual structure sheaf:

$$\hat{\mathscr{O}}_{\rm vir} = \mathscr{O}_{\rm vir} \otimes \mathscr{K}^{1/2}_{\rm vir} \ldots,$$

  where $\mathscr{K}_{\rm vir} = \det^{-1} T^{\rm vir} QM^d$ is the virtual canonical bundle.

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Pushforwards and degeneration formula

How to degenerate curve in a suitable way?

Avoiding singularities $\rightarrow$ Degeneration formula:

$$\chi(QM(\mathcal{C}_\epsilon \rightarrow N_{k,n}), \hat{\mathcal{O}}_{\mathrm{vir}} z^d) = (\mathbf{G}^{-1}\mathrm{ev}_{1,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^d), \mathrm{ev}_{2,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^d))$$

Here pairing $(\mathcal{F}, \mathcal{G}) := \chi(\mathcal{F} \otimes \mathcal{G} \otimes K^{-1/2})$,

$$\mathrm{ev}_i : QM(\mathcal{C}_{0,i} \rightarrow N_{k,n})_{\text{relative gluing point}} \rightarrow N_{k,n}$$

so that $\mathbf{G}$ is a gluing operator :

$$\mathbf{G} = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_1,p_2,*}\left(QM_{\text{relative } p_1, p_2}, \hat{\mathcal{O}}_{\mathrm{vir}}\right) \in K_T(N_{k,n})^{\otimes 2}[[z]]$$

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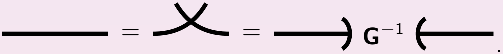

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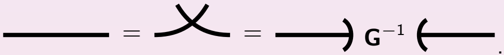

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Quantum multiplication, quantum tautological classes

We define the commutative and associative quantum product by means of the following element in $K_T(N_{k,n})^{\otimes^2}[[z]]$:

$$\mathcal{F}\circledast\cdot = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_1,p_3,*}\left(QM^d_{\mathrm{relative}\ p_1,p_2,p_3}, \mathrm{ev}^*_{\mathrm{p}_2}(\mathbf{G}^{-1}\mathcal{F})\hat{\mathcal{O}}_{\mathrm{vir}}\right)\mathbf{G}^{-1}$$

represented by

$QK_T(N_{k,n}) = K_T(N_{k,n})[[z]]$ is a unital algebra, so that:

$$\hat{\mathbf{1}}(z) = \quad 1\bullet\!\!\longrightarrow = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\left(QM^d{}_{\mathrm{relative\ p_2}}, \hat{\mathcal{O}}_{\mathrm{vir}}\right)$$

Similarly, one defines quantum tautological classes:

$$\hat{\tau}(z) = \quad \tau\bullet\!\!\longrightarrow = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\left(QM^d{}_{\mathrm{relative}\ p_2}, \hat{\mathcal{O}}_{\mathrm{vir}}\tau(\mathscr{V}|_{p_1})\right)$$

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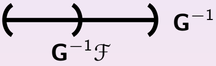

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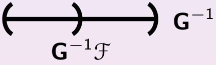

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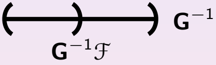

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Vertices and Fusion operator

Let us talk about $G = T \times \mathbb{C}^{\times}_{q}$-equivariant K-theory.

- Vertex, a class in $K_G(N_{k,n})_{loc}[[z]]$:

$$V^{(\tau)}(z) = $$ $$= \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\Big(QM^d{}_{\text{nonsing } p_2}, \hat{\mathscr{O}}_{\text{vir}}\tau(\mathscr{V}|_{p_1})\Big)$$

singular in $q \to 1$ limit.

- Capped Vertex, a class in $K_G(N_{k,n})[[z]]$:

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Therefore, $\lim_{q\to 1} \hat{V}^{(\tau)}(z) = \hat{\tau}(z)$

Fusion operator is defined as the following class in $K_G^{\otimes 2}(N_{k,n})_{loc}[[z]]$:

$$\Psi(z) = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_1,p_2,*}\Big(QM^d{}_{\substack{\text{relative } p_1 \\ \text{nonsing } p_2}}, \hat{\mathscr{O}}_{\text{vir}}\Big)$$

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Let us talk about $G = T \times \mathbb{C}_q^\times$-equivariant K-theory.

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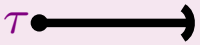
$$\hat{V}^{(\tau)}(z) = $$  $$= \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\Big(QM^d{}_{\text{relative } p_2}, \hat{\mathscr{O}}_{\text{vir}} \tau(\mathscr{V}|_{p_1}) \Big)$$

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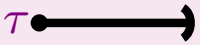

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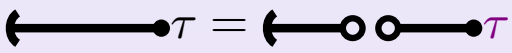

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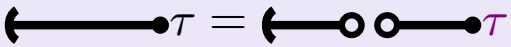

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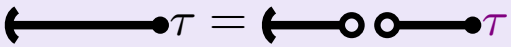

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K-theory ring

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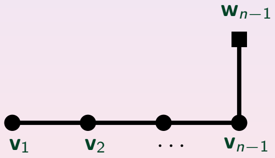

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Further directions

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