Quantum Integrable Systems via Quantum K-theory

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We will talk about the relationship between two seemingly independent areas of mathematics:

- **Quantum Integrable Systems**
  
  Exactly solvable models of statistical physics: spin chains, vertex models

  1930s: Hans Bethe: **Bethe ansatz** solution of Heisenberg model

  1960-70s: R.J. Baxter, C.N. Young: **Yang-Baxter equation**, **Baxter operator**

  1980s: Development of "QISM" by Leningrad school leading to the discovery of quantum groups by Drinfeld and Jimbo

  Since 1990s: textbook subject and an established area of mathematics and physics.

- **Enumerative geometry: quantum K-theory**
  
  Generalization of quantum cohomology in the early 2000s by A. Givental, Y.P. Lee and collaborators. Recently big progress in this direction by A. Okounkov and his school.
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▶ First hints: work of Nekrasov and Shatashvili on 3-dimensional gauge theories, now known as Gauge-Bethe correspondence:


▶ Subsequent work in geometric representation theory:

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**Understanding (enumerative) geometry of symplectic resolutions:**

"Lie algebras of XXI century" (A. Okounkov' 2012)

Important examples: Springer resolution, Hilbert scheme of points in the plane, Hypertoric varieties,...

A large class of symplectic resolutions is provided by Nakajima quiver varieties (simplest subclass: $T^* Gr(k,n)$)

In this talk our main example will be $T^* Gr(k,n)$ and more generally, cotangent bundles to (partial) flag varieties.
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Based on:


- Peter Koroteev, Anton M. Zeitlin, *Difference Equations for K-theoretic Vertex Functions of Type-A Nakajima Varieties* arXiv:1802.04463

and to some extent on

- Peter Koroteev, Daniel S. Sage, Anton M. Zeitlin, *(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality* arXiv:1811.09937
Outline

Quantum groups and quantum integrability

Nekrasov-Shatashvili ideas

Quantum K-theory and integrability

Back to Givental’s ideas + further directions
Let us consider Lie algebra $\mathfrak{g}$.

The associated \textit{loop algebra} is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and $t$ is known as \textit{spectral parameter}.

The following representations, known as \textit{evaluation modules} form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- $V_i$ are representations of $\mathfrak{g}$
- $a_i$ are values for $t$
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Quantum groups

Quantum group

\[ U_{\hbar}(\hat{g}) \]

is a deformation of \( U(\hat{g}) \), with a nontrivial intertwiner \( R_{V_1, V_2}(a_1/a_2) \):

\[ V_1(a_1) \otimes V_2(a_2) \]

\[ \quad \leftrightarrow \quad \]

\[ V_2(a_2) \otimes V_1(a_1) \]

which is a rational function of \( a_1, a_2 \), satisfying Yang-Baxter equation:

The generators of \( U_{\hbar}(\hat{g}) \) emerge as matrix elements of \( R \)-matrices (the so-called FRT construction).
Integrability and Baxter algebra

Source of integrability: commuting \textit{transfer matrices}, generating \textit{Baxter algebra} which are weighted traces of

\[
\tilde{R}_{W(u), H_{\text{phys}}} : W(u) \otimes H_{\text{phys}} \to W(u) \otimes H_{\text{phys}}
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Baxter algebra and Integrability

Source of integrability: commuting transfer matrices, generating Baxter algebra which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{phys} \rightarrow W(u) \otimes \mathcal{H}_{phys}$$

over auxiliary $W(u)$ space:

$$T_{W(u)} = \text{Tr}_{W(u)} \left( (Z \otimes 1) \tilde{R}_{W(u),\mathcal{H}_{phys}} \right)$$

Here $Z \in e^{\mathfrak{h}}$, where $\mathfrak{h} \in \mathfrak{g}$ are diagonal matrices.
Integrability:

\[ [T_{W'}(u'), T_W(u)] = 0 \]

There are special transfer matrices is called \textit{Baxter Q-operators}. Such operators generate all Baxter algebra.

Primary goal for physicists is to \textit{diagonalize} \{\(T_W(u)\)\} simultaneously.
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Primary goal for physicists is to diagonalize \( \{ T_W(u) \} \) simultaneously.
Textbook example (and main example in this talk) is XXZ Heisenberg spin chain:

$$H_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

States:

$$↑↑↑↑↓↑↑↑↓↑↑↑↑↓↑↑↑↑$$

Here $\mathbb{C}^2$ stands for 2-dimensional representation of $U_h(\hat{sl}_2)$.

Algebraic method to diagonalize transfer matrices:

Algebraic Bethe ansatz

as a part of Quantum Inverse Scattering Method developed in the 1980s.
$g = \mathfrak{sl}(2)$: XXZ spin chain

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The eigenvalues are generated by symmetric functions of Bethe roots \( \{x_i\} \):

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\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z \hbar^{-n/2} \prod_{j=1}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}, \quad i = 1 \cdots k,
\]

so that the eigenvalues \( \Lambda(u) \) of the \( Q \)-operator are the generating functions for the elementary symmetric functions of Bethe roots:

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\Lambda(u) = \prod_{i=1}^{k} (1 + u \cdot x_i)
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A real challenge is to describe representation-theoretic meaning of \( Q \)-operator for general \( g \) (possibly infinite-dimensional).
Bethe equations and Q-operator

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Modern way of looking at Bethe ansatz: solving \textbf{q-difference equations} for

\[ \Psi(z_1, \ldots, z_k; a_1, \ldots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)[[z_1, \ldots, z_k]] \]

known as

\textbf{Quantum Knizhnik-Zamolodchikov} (aka Frenkel-Reshetikhin) equations:

\[ \Psi(qa_1, \ldots, a_n, \{z_i\}) = (Z \otimes 1 \otimes \cdots \otimes 1)R_{V_1,V_n} \cdots R_{V_1,V_2} \Psi \]

+ commuting difference equations in \( z \) – variables

Here \( \{z_i\} \) are the components of twist variable \( Z \).

The latter series of equations are known as \textbf{dynamical equations}, studied by Etingof, Felder, Tarasov, Varchenko, …

In \( q \to 1 \) limit we arrive to an eigenvalue problem. Studying the asymptotics of the corresponding solutions we arrive to Bethe equations and eigenvectors.
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Nekrasov-Shatashvili ideas

In 2009 Nekrasov and Shatashvili looked at 3d SUSY gauge theories on $\mathbb{C} \times S^1$:

\[ G = U(v_1) \times U(v_2) \times \ldots U(v_{\text{rank } g}), \]

with gauge group

and some ”matter fields” (sections of associated vector $G$-bundles), to be specified below.

The collection $\{v_i\}$ determines the weights of the corresponding subspace in $\mathcal{H}$.

In the simplest case of $g = \mathfrak{sl}(2)$ we just have one $U(v)$ and

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Full Gauge/Bethe correspondence dictionary

Gauge group $G$: $U(v_1) \times U(v_2) \times \ldots \times U(v_{\text{rank}g})$

The set $\{v_i\}$ determines the weight (e.g. number of inverted spins)

Maximal torus: $\{x_{i_1}, \ldots, x_{i_{v_i}}\}$ — these are Bethe roots variables.

Matter Fields: affine space $\mathcal{M}$

- Standard matter fields: $\bigoplus_{i=1}^{\text{rank}g} V_i^* \otimes W_i$, s.t. $\text{dim}(V_i) = v_i$;

  $W_i$ is a framing ("flavor") space, where $\mathbb{C}_{a_1}^\times \times \mathbb{C}_{a_2}^\times \times \ldots$ act.
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- "**Bifundamental"** quiver data:

  $\bigoplus_{i \rightarrow j} V_i^* \otimes V_j$

  The quiver serves as a "kind of" Dynkin diagram for $g$.

To have enough supersymmetries $\bigoplus$ duals: $T^* \mathcal{M}$. 
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- “Bifundamental” quiver data:

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Moduli of Higgs vacua $\leftrightarrow$ Nakajima quiver variety:

$$T^* \mathcal{M}/\!/G = \mu^{-1}(0)/\!/G = N$$

where $\mu = 0$ is a momentum map (low energy configuration) condition.

In the case of quiver with one vertex and one framing:

$$N = T^* \text{Gr}(v, w)$$
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equivariant K-theory of Nakajima variety.

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Physicists interested in computing SUSY indices:

\[
\text{str}(e^{-\beta \phi^2} A) = \text{tr}_{\text{Ker} \phi_{\text{even}}} (A) - \text{tr}_{\text{Ker} \phi_{\text{odd}}} (A) = \text{str}_{\text{index}} \phi(A)
\]

Mathematically those correspond to (very similar to GW curve counting!) weighted K-theoretic counts of quasimaps:

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\mathcal{C} \xrightarrow{\text{quasimap } f} \text{Nakajima variety } N
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One can think of **quantum K-theory ring**:
Nekrasov and Shatashvili:

Quantum K – theory ring of Nakajima variety =

symmetric polynomials in $x_{ij} /$ Bethe equations
Key Ideas

Nekrasov and Shatashvili:

Quantum $K$ – theory ring of Nakajima variety $=$

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Input by Okounkov:

$q$ – difference equations $= qKZ$ equations $+$ dynamical equations
In the following we will talk about this in the simplest case:

- Nakajima variety: \( N = T^* Gr(k, n) \)
- Quantum Integrable System: \( \mathfrak{sl}(2) \) XXZ spin chain.
\[ T^* \text{Gr}(k, n) = N_{k,n}, \quad \sqcup_k N_{k,n} = N(n). \]

As a Nakajima variety:

\[ N_{k,n} = T^* M \sslash \!/ GL(V) = \mu^{-1}(0)_s / GL(V), \]

where

\[ T^* M = \text{Hom}(V, W) \oplus \text{Hom}(W, V) \]

Tautological bundles:

\[ \mathcal{V} = T^* M \times V \sslash \!/ GL(V), \quad \mathcal{W} = T^* M \times W \sslash \!/ GL(V) \]

For any \( \tau \in K_{GL(V)}(\cdot) = \Lambda(x_1^{\pm1}, x_2^{\pm1}, \ldots x_k^{\pm1}) \) we introduce a tautological bundle:

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Tori, Fixed points and Bethe roots

**Torus action:**

\[ A = \mathbb{C}^\times_{a_1} \times \cdots \times \mathbb{C}^\times_{a_n} \ominus W, \]

Full torus: \( T = A \times \mathbb{C}^\times_\hbar \), where \( \mathbb{C}^\times_\hbar \) scales cotangent directions

**Fixed points:** \( p = \{s_1, \ldots, s_k\} \in \{a_1, \ldots, a_n\} \)

Denote \( \mathcal{A} := \mathbb{Q}(a_1, \ldots, a_n, \hbar) \), \( R := \mathbb{Z}(a_1, \ldots, a_n, \hbar) \), then localized K-theory is:

\[
K_T(N(n))_{loc} = K_T(N(n)) \otimes_R \mathcal{A} = \sum_{k=0}^{n} K_T(N_{k,n}) \otimes_R \mathcal{A}
\]

is a \( 2^n \)-dimensional \( \mathcal{A} \)-vector space (Hilbert space for spin chain), spanned by \( O_p \).

**Classical Bethe equations:** The eigenvalues of the operators of multiplication by \( \tau \) are \( \tau(x_1, \cdots, x_k) \) evaluated at the solutions of the following equations:

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Quantum tautological classes and Bethe equations

We will use theory of quasimaps:

$$\mathbb{C} \longrightarrow \mathcal{N}_{k,n}$$

in order to deform tensor product: $A \otimes B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B z^d$.

We will also define quantum tautological classes:

$$\hat{\tau}(z) = \tau + \sum_{d=1}^{\infty} \tau_d z^d \in K_T(N(n))[z]$$

**Theorem.** [P. Pushkar, A. Smirnov, A.Z] The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1, \ldots, x_k)$ evaluated at the solutions of the following equations:

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The quantum K-theoretic meaning of the Q-operator

**Theorem.** [P. Pushkar, A. Smirnov, A.Z.]

- The quantum multiplication on quantum tautological class corresponding to $\tau_u := \bigoplus_{m \geq 0} u^m \Lambda^m \mathcal{V}$ coincides with $Q$-operator, i.e.

  $$\hat{\tau}_u(z) = Q(u)$$

- Explicit universal formulas for quantum products:

  $$\hat{\Lambda}^\ell \mathcal{V}(z) = \Lambda^\ell \mathcal{V} + a_1(z) F_0 \Lambda^{\ell-1} \mathcal{V} E_{-1} + \cdots + a_\ell(z) F_\ell E_{-1},$$

where $a_m(z) = \frac{(h-1)^m}{(m)_h} \frac{m(m+1)}{2} \frac{K^m}{\prod_{i=1}^m (1-(−1)^n z^{-1} h^i K)},$

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Quasimaps

**Quasimap** \( f : \mathbb{C} \rightarrow N_{k,n} \) is the following collection of data:

- vector bundle \( \mathcal{V} \) on \( \mathbb{C} \) of rank \( k \).
- section \( f \in H^0(\mathbb{C}, \mathcal{M} \oplus \mathcal{M}^* \otimes \hbar) \), satisfying the condition \( \mu = 0 \), where \( \mathcal{M} = \text{Hom}(\mathcal{V}, \mathcal{W}) \), so that \( \mathcal{W} \) is a trivial bundle of rank \( n \).

\[
e_{\nu_p}(f) = f(p) \in [\mu^{-1}(0)/GL(V)] \supset N_{k,n}
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Quasimap is **stable** if \( f(p) \in N_{k,n} \) for all but finitely many points, known as **singularities** of quasimap.

For the moduli space of quasimaps

\[
QM(N_{k,n}) = \text{stable quasimaps to } N_{k,n}/\sim
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only \( \mathcal{V} \) and \( f \) vary, while \( \mathbb{C} \) and \( \mathcal{W} \) remain the same.

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deg(f) := deg(\mathcal{V}), \quad QM(N_{k,n}) = \sqcup_{d \geq 0} QM^d(N_{k,n}).
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e_{\nu}(f) = f(p) \in [\mu^{-1}(0)/\text{GL}(V)] \supset N_{k,n}
\]

Quasimap is **stable** if \( f(p) \in N_{k,n} \) for all but finitely many points, known as **singularities** of quasimap.

For the moduli space of quasimaps

\[
QM(N_{k,n}) = \text{stable quasimaps to } N_{k,n}/\sim
\]

only \( \mathcal{V} \) and \( f \) vary, while \( C \) and \( \mathcal{W} \) remain the same.

\[
deg(f) := deg(\mathcal{V}), \quad QM(N_{k,n}) = \bigsqcup_{d \geq 0} QM^d(N_{k,n}).
\]
Quasimaps

**Quasimap** $f: \mathbb{C} \dashrightarrow N_{k,n}$ is the following collection of data:

- vector bundle $\mathcal{V}$ on $\mathbb{C}$ of rank $k$.
- section $f \in H^0(\mathbb{C}, \mathcal{M} \oplus \mathcal{M}^* \otimes \hbar)$, satisfying the condition $\mu = 0$, where $\mathcal{M} = \text{Hom}(\mathcal{V}, \mathcal{W})$, so that $\mathcal{W}$ is a trivial bundle of rank $n$.

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Relative quasimaps

Resolution, to make evaluation map proper:

\[
QM^d(N_{k,n})_{\text{relative } p} \xrightarrow{\sim \text{ev}_p} QM^d(N_{k,n})_{\text{nonsing } p} \xrightarrow{\text{ev}_p} N_{k,n}
\]
Relative quasimaps

Resolution, to make evaluation map proper:

\[ QM^d(N_{k,n})_{\text{relative } p} \xrightarrow{\text{ev}_p} \tilde{\text{ev}}_p \]

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Relative quasimaps

Resolution, to make evaluation map proper:

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\[ \tilde{ev}_p \]

\[ QM^d(N_{k,n})_{\text{nonsing } p} \]

\[ ev_p \]

\[ N_{k,n} \]

That allows the curve to break: emergence of “\textit{accordeons}”:

\[ p' \rightarrow C' \xrightarrow{f'} N_{k,n} \]

\[ \pi \]

\[ p \rightarrow C \]

\( i \)\( \pi \) is a stabilization of \((C', p')\)

\( ii \)\( f' \): nonsing at \( p' \) and nodes of \( C' \)

\( iii \)\( \text{Aut}(f') \) is finite
Virtual sheaves

$QM^d(N_k,n)$ have perfect deformation-obstruction theory:

- If $(\mathcal{V}, \mathcal{W})$ defines quasimap nonsingular at $p$,

  $$T_{(\mathcal{V}, \mathcal{W})}^{\text{vir}} QM^d_{\text{nonsing } p}(N_k,n) = \text{Def} - \text{Obs} = H^\bullet(\mathcal{P} \oplus \hbar \mathcal{P}^*),$$

  where $\mathcal{P}$ is the polarization bundle on the curve $\mathcal{C}$:

  $$\mathcal{P} = \mathcal{W} \otimes \mathcal{V}^* - \mathcal{V}^* \otimes \mathcal{V}.$$

- Virtual structure sheaf:

  $$\hat{\mathcal{O}}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \mathcal{K}_{\text{vir}}^{1/2} \ldots,$$

  where $\mathcal{K}_{\text{vir}} = \det^{-1} T_{\text{vir}}^{\text{vir}} QM^d$ is the virtual canonical bundle.
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$$\mathcal{P} = W \otimes V^* - V^* \otimes V.$$

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Pushfowards and degeneration formula

How to degenerate curve in a suitable way?

Avoiding singularities → Degeneration formula:

\[ \chi(QM(C_\epsilon \to N_{k,n}), \hat{\text{vir}} z^d) = (G^{-1} ev_1, * (\hat{\text{vir}} z^d), ev_2, * (\hat{\text{vir}} z^d)) \]

Here pairing \((\mathcal{F}, \mathcal{G}) := \chi(\mathcal{F} \otimes \mathcal{G} \otimes K^{-1/2}),\)

\[ ev_i : QM(C_{0,i} \to N_{k,n})_{\text{relative gluing point}} \to N_{k,n} \]

so that \(G\) is a gluing operator:

\[ G = \sum_{d=0}^{\infty} z^d ev_{p_1, p_2, *} \left( QM_{\text{relative } p_1, p_2}, \hat{\text{vir}} \right) \in K_T(N_{k,n}) \otimes^2 [[z]] \]
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Quantum multiplication, quantum tautological classes

We define the commutative and associative quantum product by means of the following element in $K_T(N_k,n)^\otimes 2[[z]]$:

$$F \otimes \cdot = \sum_{d=0}^{\infty} z^d \text{ev}_{p_1,p_3,*} \left( QM^d_{\text{relative } p_1,p_2,p_3} \exp_p^* \left( G^{-1}F \right) \hat{O}_\text{vir} \right) G^{-1}$$

represented by

$$\left( \begin{array}{c}
G^{-1}F \\
\end{array} \right) G^{-1}$$

$QK_T(N_k,n) = K_T(N_k,n)[[z]]$ is a unital algebra, so that:

$$\hat{1}(z) = 1 \longrightarrow \sum_{d=0}^{\infty} z^d \text{ev}_{p_2,*} \left( QM^d_{\text{relative } p_2} \hat{O}_\text{vir} \right)$$

Similarly, one defines quantum tautological classes:

$$\hat{\tau}(z) = \tau \longrightarrow \sum_{d=0}^{\infty} z^d \text{ev}_{p_2,*} \left( QM^d_{\text{relative } p_2} \hat{O}_\text{vir} \tau(\mathcal{V} |_{p_1}) \right)$$
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represented by

$$
\begin{pmatrix}
\text{\text{G}^{-1}\mathcal{F}} \\
\hline
\end{pmatrix}
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$$

represented by

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\begin{array}{c}
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\hline \hline \mathcal{F} \\
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$$
Let us talk about $G = T \times \mathbb{C}_{q}^{	imes}$-equivariant K-theory.

- **Vertex**, a class in $K_{G}(N_{k,n})_{loc}[[z]]$:

$$V^{(\tau)}(z) = \tau = \sum_{d=0}^{\infty} z^{d} \text{ev}_{p_{2}}(Q^{d}_{\text{nonsing}}p_{2}, \hat{O}_{\text{vir}}\tau(V|_{p_{1}}))$$

singular in $q \to 1$ limit.

- **Capped Vertex**, a class in $K_{G}(N_{k,n})[[z]]$:

$$\hat{V}^{(\tau)}(z) = \tau = \sum_{d=0}^{\infty} z^{d} \text{ev}_{p_{2}}(Q^{d}_{\text{relative}}p_{2}, \hat{O}_{\text{vir}}\tau(V|_{p_{1}}))$$

Therefore, $\lim_{q \to 1} \hat{V}^{(\tau)}(z) = \hat{\tau}(z)$

**Fusion operator** is defined as the following class in $K_{G}^{\otimes 2}(N_{k,n})_{loc}[[z]]$:

$$\Psi(z) = \sum_{d=0}^{\infty} z^{d} \text{ev}_{p_{1}, p_{2}}(Q^{d}_{\text{nonsing}}p_{2}, \hat{O}_{\text{vir}})$$
Vertices and Fusion operator

Let us talk about \( G = T \times \mathbb{C}_q^\times \)-equivariant K-theory.

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\]

Therefore, \( \lim_{q \to 1} \hat{V}^{(\tau)}(z) = \hat{\tau}(z) \)

**Fusion operator** is defined as the following class in \( K_G^2(N_k,n)_{\text{loc}}[[z]] \):

\[
\Psi(z) = \sum_{d=0}^{\infty} z^d \text{ev}_{p_1,p_2,*} \left( QM^d_{\text{relative} p_1, \text{nonsing} p_2}, \hat{\mathcal{O}}_{\text{vir}} \right)
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Fusion relates two types of vertices:

\[ \hat{V}^{(\tau)}(z) = \Psi(z) \bar{V}^{(\tau)}(z) \]

**Theorem.** i) [A. Okounkov] Fusion operator satisfies q-difference equation:

\[
\Psi(qz) = M(z)\Psi(z)\mathcal{O}(1)^{-1},
\]

where \(\mathcal{O}(1)\) is the operator of classical multiplication by the corresponding line bundle and

\[
M(z) = \sum_{d=0}^{\infty} z^d ev_*(QM^d_{\text{relative } p_1, p_2} \cdot \hat{O}_{\text{vir}} \det H^* (\mathcal{V} \otimes \pi^*(\mathcal{O}_{p_1}))) G^{-1},
\]

where \(\pi\) is a projection from semistable curve \(\mathcal{C}' \to \mathcal{C}\) and \(\mathcal{O}_{p_1}\) is a class of point \(p_1 \in \mathcal{C}\).

ii) [P. Pushkar, A. Smirnov, A.Z] Under the specialization \(q = 1\) the operator \(M(z)\) coincides with the operator of quantum multiplication by the quantum line bundle:

\[
M(z)|_{q=1} = \mathcal{O}(1)(z) \otimes .
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Theorem. i) [A. Okounkov] Fusion operator satisfies q-difference equation:

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\[ M(z) = \sum_{d=0}^{\infty} z^d \text{ev}_* \left( QM_{\text{relative}}^{d}p_1,p_2, \hat{\Omega}_{\text{vir}} \det H^\bullet\left( V \otimes \pi^* (\mathcal{O}_{p_1}) \right) \right) G^{-1}, \]

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q-difference equation

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\[ M(z) = \sum_{d=0}^{\infty} z^d \text{ev}_* \left( QM^d_{\text{relative } p_1, p_2} \hat{O}_{\text{vir}} \det H^* \left( \mathcal{V} \otimes \pi^*(O_{p_1}) \right) \right) G^{-1}, \]

where \( \pi \) is a projection from semistable curve \( C' \to C \) and \( O_{p_1} \) is a class of point \( p_1 \in C \).

ii) [P. Pushkar, A. Smirnov, A.Z] Under the specialization \( q = 1 \) the operator \( M(z) \) coincides with the operator of quantum multiplication by the quantum line bundle:

\[ M(z)|_{q=1} = \hat{O}(1)(z) \otimes . \]
q-difference equation

Fusion relates two types of vertices:

\[ \hat{V}^{(\tau)}(z) = \Psi(z)V^{(\tau)}(z) \]

\[ \tau = \tau \]

**Theorem.** i) [A. Okounkov] Fusion operator satisfies q-difference equation:

\[ \Psi(qz) = M(z)\psi(z)O(1)^{-1}, \]

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**Theorem.** [P. Pushkar, A. Smirnov, A.Z.]

i) Localization formula implies the following integral formula for the vertex:

\[
V_p^{(\tau)}(z) = \frac{1}{2\pi i \alpha_p} \int_{C_p} ds_i \prod_{i=1}^{k} \frac{d s_i}{s_i} e^{-\frac{\ln(z^\#) \ln(s_i)}{\ln(q)}} \prod_{i,j=1}^{k} \varphi\left(\frac{s_i}{s_j}\right) \prod_{i=1}^{n} \prod_{j=1}^{k} \varphi\left(\frac{q}{\hbar} \frac{s_i}{a_i}\right) \tau(s_1, \ldots, s_k),
\]

where \(\varphi(x) = \prod_{i=0}^{\infty} (1 - q^i x)\), \(z^\# = (-1)^n \hbar^{n/2} z\), \(\alpha_p\) is a normalization parameter.

ii) The eigenvalues \(\tau_p(z)\) of \(\hat{\tau}(z)\) are labeled by fixed points are given by the following formula:

\[
\tau_p(z) = \lim_{q \to 1} \frac{V_p^{(\tau)}(z)}{V_p^{(1)}(z)} = \tau(x_{i_1}, x_{i_2}, \ldots, x_{i_k})
\]

where \(V_p^{(\tau)}(z)\) are the components of bare vertex in the basis of fixed points and \(\{x_{i_r}\}\) are the solutions of Bethe equations.
K-theory ring

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Cotangent bundle to partial flag variety is a Nakajima variety of type A:

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Further directions

- There are quantum Wronskian relations, which Q-operators satisfy (\(\tilde{\mathcal{Q}}Q\)-system). Geometric meaning?

  Recent answer is given in terms of q-opers by P. Koroteev, D. Sage, A. Zeitlin. Enumerative meaning?

- Enumerative geometry of symplectic resolutions \(\rightarrow\) new kinds of integrable systems. The simplest example: Hilbert scheme of points on a plane.

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Thank you!