Vibration of Drumhead: Sizes in the Music

Chelsey Fontenot

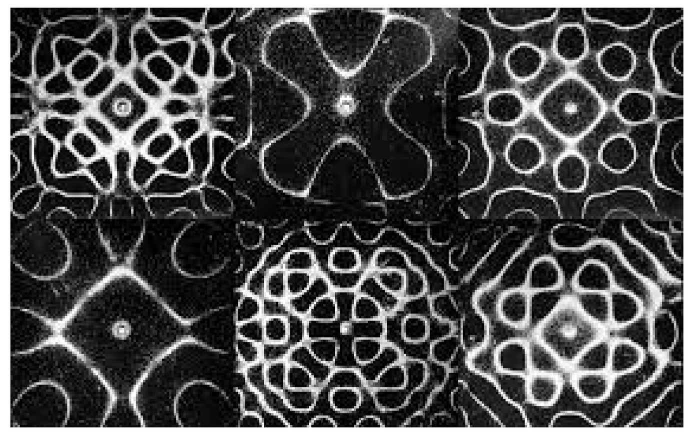
Department of Mathematics at Louisiana State University

Introduction

Chladni Pattern models the nodal patterns in vibrating plates. Ernst Chladni (Father of acoustics, 17561827) introduced the Chlandi Pattern using sand and metal plates. By striking the plates, the sand would attract to areas where no vibration occurred, which leads to strange patterns forming.

Our Research Questions

To dive deeper into the nodal patterns, we decided to focus on two important research questions which include: How many nodal lines or curves are created and how many nodal lines or curves intersect the boundary? This project focuses on the vibration of circular drumhead and answers the questions discussed above.



Playing Guitar

The vibration of guitar string is represented by the ordinary differential equation

$$\begin{cases} -y''(x) = \lambda^2 y(x) & x \in (0,1), \\ y(0) = y(1) = 0, \end{cases}$$

where λ is the frequency. Solving the equation by trying the solutions $y(x) = e^{rx}$, we arrive at

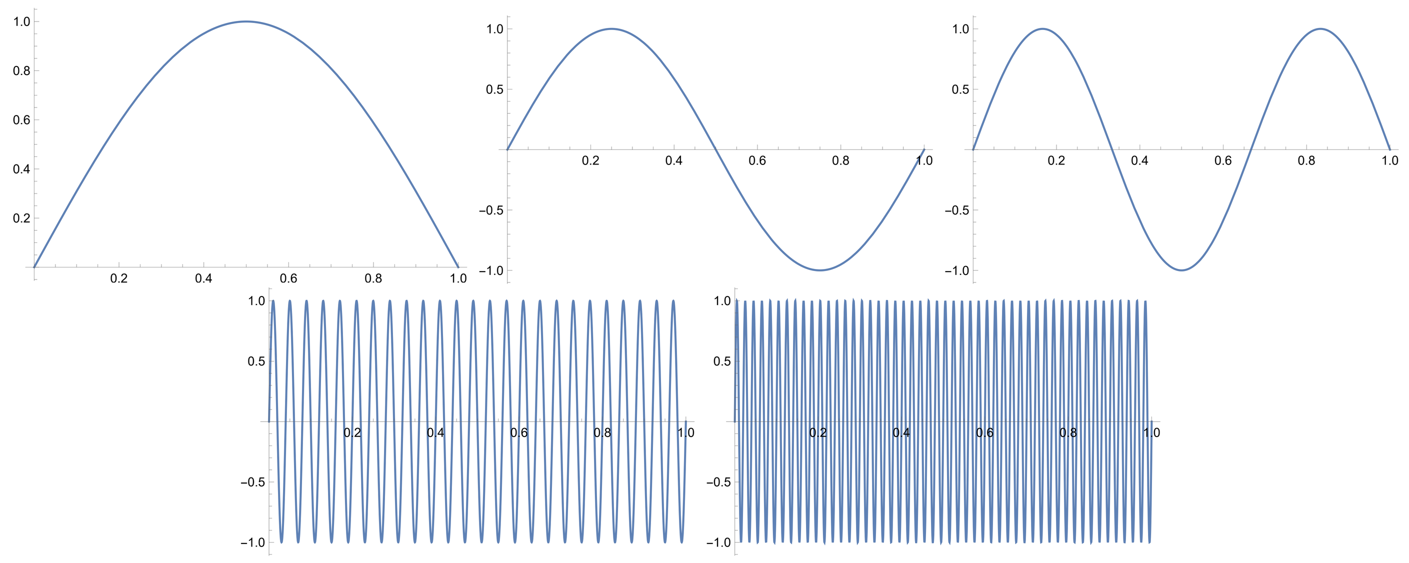
$$y(x) = c_1 cos(\lambda x) + c_2 sin(\lambda x). \tag{1}$$

The boundary conditions lead to $y(x) = c_2 sin(k\pi x) = 0$ in $[0, 1]$ for eigenvalues $\lambda = k$ and $k = 1, 2, 3, ... \to \infty$. The specific nodal points $sin(k\pi x) = 0$ are

$$x = \frac{1}{k}, \frac{2}{k}, \frac{3}{k}, ..., \frac{k-1}{k}.$$

The number of nodal points are

$$H^0\{x \in (0,1) | y(x) = 0\} = k - 1 \le C\lambda. \tag{2}$$



Chlandi Pattern: Playing Guitar for k = 1, 2, 3,50,100

Two-Dimensional Circular drumhead and its Solutions

The vibration of drumhead is model by the Dirichlet eigenvalue problems

$$\begin{cases} -\Delta u(z) = \lambda^2 u(z) & z \in \Omega, \\ u(z) = 0 & z \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. It is interesting and fundamental to study the sizes of nodal sets in the vibration. It was conjectured by the Fields medalist Shing-Tung Yau [1] that the sizes of interior nodal sets is bounded as

$$c\lambda \le H^{n-1}\{z \in \Omega | u(z) = 0\} \le C\lambda. \tag{3}$$

The interior nodal sets touch the boundary $\partial\Omega$ to form boundary nodal sets. It was shown by Zhu [2] that the sharp upper bound of boundary critical sets is

$$H^{n-2}\{z \in \partial\Omega | |\nabla u(z)| = 0\} \le C\lambda. \tag{4}$$

Our goal is to verify that the upper bound in (3) and (4) are optimal. We exam the vibration of two-dimensional Chlandi Pattern in the ball

$$\begin{cases} -\Delta u(x,y) = \lambda^2 u(x,y) & (x,y) \in \mathbb{B}_1 \\ u(x,y) = 0 & (x,y) \in \partial\mathbb{B}_1 \end{cases}$$

where $\mathbb{B}_1 = \{(x,y) | x^2 + y^2 < 1\}$ and $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. To solve this equation, we used polar coordinates (r, θ) where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Utilizing our boundary condition, we began to solve the two-dimensional aspect concerning a disc shaped:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. \tag{5}$$

The next step in solving for the nodal lines and curves is to use the separation of variables method as shown below:

$$u(r, \theta) = R(r)\Phi(\theta)$$

We simplified the original equation by separating the variables given in the equation. We can now substitute this value into the original equation to give us Bessel's Equation:

$$r^2 R''(r) + rR'(r) + (\lambda^2 r^2 - k^2)R(r) = 0 \tag{6}$$

$$-\Phi''(\theta) = k^2 \Phi(\theta)$$

The equation (6) is called Bessel's Equation. The solution of (6) is given as the $k-$th Bessel function $J_k(\lambda r)$. The solution is as follows:

$$u(r, \theta) = A_1 J_k(j_{k,s} r) sin(k\theta + \theta_0) \quad or \quad u(r, \theta) = A_2 J_k(j_{k,s} r) cos(k\theta + \theta_0)$$

where $\lambda = j_{k,s}$ is the root of $J_k(\lambda) = 0$.

Study of Nodal curves

If $sin(k\theta) = 0$, then $\theta = \frac{m\pi}{k}$ with $m = 0, 1, \cdots, \frac{2k-1}{k}$. Thus the nodal sets of $u(r, \theta)$ is a collection of and $s-1$ concentric circles with radius $\frac{j_{k,l}}{j_{k,s}}$, $l = 1, \cdots, s-1$.

The length of nodal line from k diameters is $2k$.

To study the nodal $s-1$ concentric circles, we consider the following two cases:

1. *Case 1:* k tends to infinity and s is bounded
2. *Case 2:* s tends to infinity and k is bounded

Case 1: As $k \to \infty$ and $s \le M$ for positive constant M, $\lambda = j_{k,s} \to \infty$. Using the fact that

$$j_{k,M} = k + o(k) \quad \text{as } k \to \infty, \tag{7}$$

$$H^1\{(x,y) \in \mathbb{B}_1 | u(x,y) = 0\} = 2k + 2\pi \sum_{l=1}^{s-1} \frac{j_{k,l}}{j_{k,s}} \ge 2k \approx C\lambda. \tag{8}$$

Case 2: As $s \to \infty$ and $k \le M$ for positive constant M, $\lambda = j_{k,s} \to \infty$. Using the fact that

$$j_{M,s} = (s + \frac{M}{2} - \frac{1}{4})\pi + O(s^{-1}) \text{ as } s \to \infty, \tag{9}$$

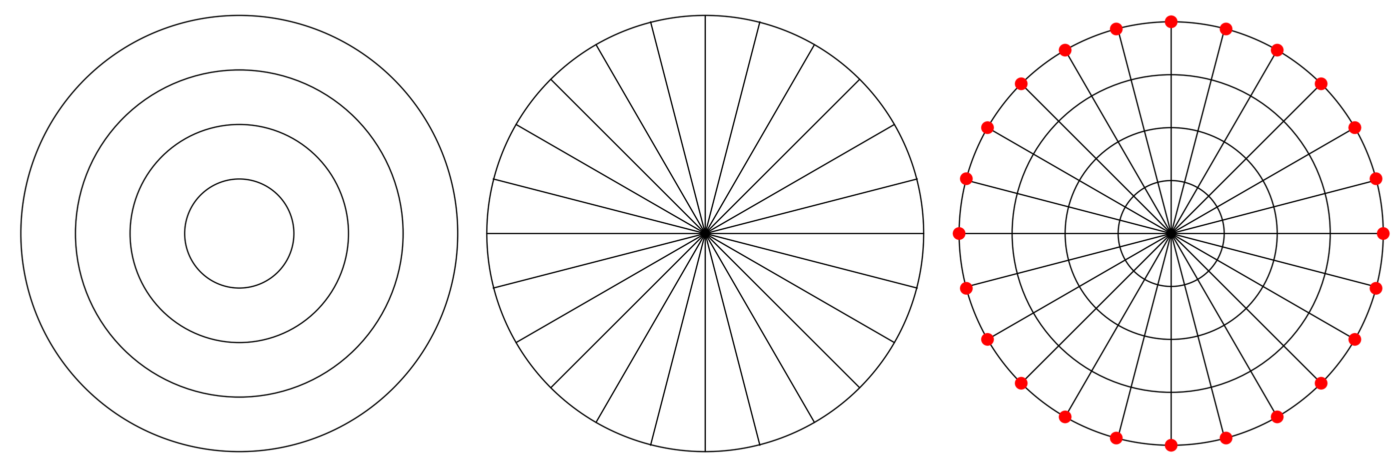
$$\begin{aligned} H^1\{(x,y) \in \mathbb{B}_1 | u(x,y) = 0\} &= 2k + 2\pi \sum_{l=1}^{s-1} \frac{j_{k,l}}{j_{k,s}} \ge 2\pi \sum_{l=\frac{s}{2}}^{s-1} \frac{j_{k,l}}{j_{k,s}} \\ &\approx 2\pi \frac{\sum_{l=\frac{s}{2}}^{s-1} (l + \frac{M}{2} - \frac{1}{4})}{(s + \frac{M}{2} - \frac{1}{4})} \\ &\approx C\lambda. \end{aligned} \tag{10}$$

From (11) and (10), we learn that the lower bounds and upper bounds in (3) are optimal, which can be achieved at the two-dimensional disc.

The intersection of the k diameters with the boundary in the ball are points $\{(x,y) \in \partial\mathbb{B}_1 | \ |\nabla u| = |u'(1) = 0\}$. From case 1,

$$H^0\{(x,y) \in \partial\mathbb{B}_1 | \ |\nabla u(x,y)| = 0\} = 2k \approx C\lambda. \tag{11}$$

This shows that the upper bounds in (4) is optimal in two-dimensional balls.



Chladni Pattern on a Drumhead

References

References

1. S.-T. Yau, Problem section, Seminar on Differential Geometry, Annals of Mathematical Studies 102, Princeton, 1982, 669706.
2. J. Zhu, Doubling inequalities and upper bounds of critical sets of Dirichlet eigenfunctions, J. Funct. Anal., 281(2021), 109155.
3. A. Savo, Lower bounds for the nodal length of eigenfunctions of the Laplacian. Ann. Global Anal. Geom. 19 (2001), no. 2, 133–151.

Acknowledgments

1. Professor Jiuyi Zhu, Department of Mathematics, LSU.