A priori estimates, existence and non-existence of positive solutions of generalized mean curvature equations

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Abstract. This paper concerns a priori estimates and existence of solutions of generalized mean curvature equation with Dirichlet boundary value conditions in smooth domain. Using the blow-up method with the Liouville-type theorem of p laplacian equation, we obtain a priori bounds and the estimates of interior gradient for all solutions. The existence of positive solution is derived by topological method. We also consider the non-existence of solutions by Pohozaev identities.

1. Introduction

We consider generalized mean curvature equation

$$\begin{cases} -\mathrm{div}((1+|Du(x)|^2)^{\frac{p}{2}-1}Du(x)) = u^q(x), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where p, q are real positive numbers, Ω is a bounded domain with smooth boundary $\partial\Omega$, $\Omega \subset \mathbb{R}^N$, $1 < p < N$, $N \geq 2$ is the dimension of Ω. We call $-\mathrm{div}((1+|Du(x)|^2)^{\frac{p}{2}-1}Du(x))$ the generalized mean curvature operator. Let $p^* = \frac{Np}{N-p}$ be the critical exponent of the generalized mean curvature operator. We focus on the case of $p-1 < q < p^*-1$. Let us mention the importance of this operator. In the case of $p = 1$, it turns out to be the mean curvature operator, which has been attracting much attention in differential geometry and partial differential equation. The interested readers may refer to [**GT**], [**W**] and references therein for more details. In the case of $p = 2$, it becomes the classical Laplacian operator. It has been extensively studied from many aspects such as variational approaches, a priori estimates and so on. It also has many important applications in physics, biology and

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other interdisciplines. In this paper, we mainly consider the generalized mean curvature equation from a priori estimates aspect, which further implies the existence of positive solutions. Although (1.1) has a variational structure and the existence can be obtained by maximum-minimum theorem, the solutions obtained from these methods usually have no priori bounds. The a priori estimates give more information about the solutions. Furthermore, if the variational method is not applicable, i.e. the Euler-Lagrange equation does not exist, the question of existence of solutions can be also dealt with a priori estimates and topological methods. In the literature, several approaches are widely applied to elliptic equations to derive a priori estimates. (1) The method of Rellich-Pohozaev identities together with moving planes in [**DLN**]. The solution around the boundary is estimated by moving planes method and the interior bound is estimated by Rellich-pohozaev identities and bootstrap method. (2) The rescaling or blow-up method and suitable Liouville-type theorem (see. e.g. [**GS**]). The method is carried out by contradiction by assuming that there exists a sequence of unbounded solutions. Under the appropriate rescaling and elliptic estimates, the sequences converge to a solution in the whole space or half space, which contradicts the corresponding Liouville-type theorem. We applied this idea in our proofs. (3) The method of Hardy-Sobolev inequalities, which is introduced in [**BT**]. The idea is to use first eigenfunction of Laplacian operator as multiplies, Hardy-Sobolev inequalities and bootstrap arguments to derive a uniform bound.

We study the a priori estimates of positive solutions of (1.1) and prove the uniform bound on full sets of solutions. Our main results can be stated as follows.

Theorem 1. *Let Ω be bounded smooth domain and $p-1<q<p^*-1$. Assume that $u(x)\in W_0^{1,p}(\Omega)$ is any weak solution of (1.1), then there exists M independent of u such that $\|u\|_{L^\infty(\Omega)}\leq M$.*

We also have

Corollary 1. *Let Ω be bounded smooth domain, the mean curvature of $\partial\Omega$ have positive bounds H_0 from below and $p-1<q<p^*-1$. Assume that $u\in W_0^{1,p}(\Omega)$ is a weak solution of (1.1), then u is a C^2 classical solution. Moreover, there exists a constant C independent of u such that $\|u\|_{C^2(\bar{\Omega})}\leq C$.*

Based on a priori estimates, we obtain the existence of nonnegative solutions in (1.1) by fixed point theorem.

Theorem 2. *Let Ω be bounded smooth domain, the mean curvature of $\partial\Omega$ have positive bounds H_0 from below, $p\geq 2$ and $p-1<q<p^*-1$. Then the equation (1.1) has at least one positive solution.*

Remark 1. *The above theorems hold for a more general function $f(u)$ instead of u^q. We only prove this case for simplicity.*

Furthermore, we take account of the non-existence of the nonnegative solutions of (1.1). We establish

THEOREM 3. *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^N$, which is strictly star-shaped with respect to the origin in $\mathbb{R}^N$. Then, any solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in* (1.1) *vanishes identically in the case of $q \geq p^* - 1$.*

We also show that the comparison principle for generalized mean curvature operator, which is included in a general form in [**GT**]. We prove it in a different way that is very similar to p laplacian equation for completeness. The comparison principle plays an important role in proving the boundary gradient estimates and existence theorem.

This paper is organized as follows. In Section 2, we present some preliminary knowledge, which includes Liouville-type theorems for p laplcian equation in Euclidean space and half space. Section 3 is devoting to establishing a priori estimates for every positive solution. We first obtain the priori bound by blow-up method and Liouville-type theorems. Then, we show the interior gradient estimate by the control of the distance function using a new developed rescaling method. In section 3, we derive the existence theorem of this type of quasi-linear equation following from the existence of fixed points on compact operators in a cone. The non-existence theorem is shown in section 4. In appendix, we give the proof of the comparison principle for generalized mean curvature operator. Throughout the paper, C, C_0 and M denote generical positive constants, which is independent of u and may vary from line to line.

2. Preliminary knowledge

In this section, we collect some results for quasi-linear elliptic equation and Liouville-type theorems for p laplacian equation. The generalized mean curvature equation is a typical quasi-linear equation with principal part in divergence form. Let $a_i(Du) = (1+|Du|^2)^{\frac{p}{2}-1} D_i u$ and $p_i = D_i u$. The following structure condition holds:

$$\nu(p)(1+|Du|)^{p-2}\sum_{i=1}^N \xi_i^2 \leq \sum_{i,j=1}^N \frac{\partial_i a_i(Du)}{\partial p_j}\xi_i\xi_j \leq \mu(p)(1+|Du|)^{p-2}\sum_{i=1}^N \xi_i^2, \tag{2.1}$$

where $\mu(p), \nu(p)$ are positive constants with respect to p. If $\max_\Omega |Du|$ is bounded, the generalized mean curvature equation is strictly elliptic for $1 < p < N$.

The next lemma is the Liouville-type theorem for p laplacian equation in $\mathbb{R}^N$[**SZ**].

LEMMA 1. *Assume that $p - 1 < q < p^* - 1$ and $1 < p < N$, then every nonnegative solution $u(x) \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ in (2.2) is trivial,*

$$-\Delta_p u(x) = u^q(x), \qquad x \in \mathbb{R}^N, \tag{2.2}$$

where

$$\Delta_p u = \mathrm{div}(|Du|^{p-2}Du).$$

Recently the Liouville-type theorem for p laplacian equation in half space is established in [**Z**].

LEMMA 2. *Assume that $p-1<q<p^*-1$ and $1<p<N$, then every nonnegative solution $u(x)\in W_0^{1,p}(\mathbb{R}_+^{\mathrm{N}})\cap C(\overline{\mathbb{R}_+^N})$ in (2.3) is trivial,*

$$\begin{cases} -\Delta_p u(x) = u^q(x), & x\in \mathbb{R}_+^N, \\ u(x)=0, & x\in \partial\mathbb{R}_+^N. \end{cases} \tag{2.3}$$

See also the following Liouville-type theorem in [**Z**].

LEMMA 3. *There does not exist non-negative solutions $u(x)\in W_{loc}^{1,p}(\mathbb{R}^N)\cap C(\mathbb{R}^N)$ for*

$$\Delta_p u + k \le 0, \qquad x\in \mathbb{R}_+^N,$$

where k is a positive constant.

3. A priori estimates

In this section, we will first show all the solutions in (1.1) have a uniform bound, which plays an important role when applying topological method to prove the existence. The method is based on the classical blow-up technique in ([**GS**]). We obtain that the generalized mean curvature equation will blow up into the p laplacian equation under certain rescaling.

Proof of Theorem 1. Suppose that the conclusion is not true, we argue by contradiction. Then, there exists a sequence of $u_n(x_n)=\sup_\Omega u_n(x)$ such that $M_n := u(x_n)\to\infty$ as $j\to\infty$.

Introduce a new function

$$w_n(y) := \frac{u_n(M_n^{-k}y + x_n)}{M_n}$$

in Ω_n, where $\Omega_n := M_n^k(\Omega\setminus x_n)$ and $k:=\frac{q-p+1}{p}$. By the assumption that $q>p-1$, we have $k>0$. Since $u_n(x)$ satisfies

$$\int_\Omega (1+|Du_n|^2)^{\frac{p}{2}-1} Du_n D\varphi\, dx = \int_\Omega u_n^q \varphi\, dx$$

for any $\varphi\in C_0^\infty(\Omega)$. With the aid of change of variable, w_n satisfies

$$\begin{cases} \int_{\Omega_n} (\frac{1}{M_n^{2(k+1)}} + |Dw_n|^2)^{\frac{p}{2}-1} Dw_n D\phi\, dy &= \int_{\Omega_n} w_n^q \phi\, dy, \\ \|w_n\|_{L^\infty(\Omega_n)} &\le 1, \\ w_n(0) &= 1, \\ w_n(y) &= 0 \qquad \text{on } \partial\Omega_n \end{cases} \tag{3.1}$$

for all $\phi \in C_0^\infty(\Omega_n)$, where $\phi(y) := \varphi(M_n^{-k}y + x_n)$.
Let

$$d_n := dist(x_n, \partial\Omega).$$

Two cases may occur as $n \to \infty$: either case (1)

$$M_n^k d_n \to \infty$$

for a subsequence still denoted as before. or case (2),

$$M_n^k d_n \to d$$

for a subsequence still denoted as before.

If case (1) occurs, we have $M_n^k B_{d_n}(0) \subset \Omega_n$ and $B_{M_n^k d_n}(0) \to \infty$ as $n \to \infty$. Then, for any smooth compact set D in $\mathbb{R}^N$, there exists n_0 such that $\bar{D} \subset \Omega_n$ for all $n \geq n_0$. Since $\|w_n\|_{L^\infty(\Omega_n)} \leq 1$, from the $C^{1,\alpha}$ local estimates in [**T**] with $\alpha > 0$, there exists some constant C independent of w_n and n such that

$$w_n \in C^{1,\alpha}(\bar{D}) \quad \text{and} \quad \|w_n\|_{C^{1,\alpha}(\bar{D})} \leq C, \quad \forall n \geq n_0.$$

By Arzelá-Ascoli Theorem, there exist a function $w \in C^1(\bar{D})$ and a convergent subsequence denoted as before such that $w_n \to w$ in $C^1(\bar{D})$. Furthermore, using a diagonal line argument, we obtain that $w \in C^1(\mathbb{R}^N)$ solves

$$\begin{cases} \int_{\mathbb{R}^N} |Dw|^{p-2} Dw D\phi \, dy &= \int_{\mathbb{R}^N} w^q \phi \, dy, \\ w(0) &= 1, \\ w(x) &\geq 0, \qquad \forall x \in \mathbb{R}^N, \end{cases}$$

which obviously contradicts the Liouville-type theorem in lemma 1.

Assume the case (2), $M_n^k d_n \to d$ as $n \to \infty$ for some $d \geq 0$. The domain converges to the half space $\mathbb{R}_+^N := \{x_n \geq -d\}$. Through a subsequence of w_n, by $C^{1,\alpha}$ local estimate, Arzelá-Ascoli Theorem and diagonal line argument as before, $w_n \to w$ locally in $C^1(\mathbb{R}_+^N)$ as $n \to \infty$, and $w \in C^1(\bar{\mathbb{R}}_+^N)$ satisfies

$$\begin{cases} \int_{\mathbb{R}_+^N} |Dw|^{p-2} Dw D\phi \, dy &= \int_{\mathbb{R}_+^N} w^q \phi \, dy, \\ w(0) &= 1, \\ w(x) &= 0 \qquad \text{on } \partial\mathbb{R}_+^N \end{cases}$$

for any $\phi \in C_0^\infty(\mathbb{R}_+^N)$. Under a suitable translation, it again contradicts the Liouville-type theorem in half space in lemma 2. In conclusion, we complete the proof of the theorem. □

Next we will prove the interior gradient estimate for the solutions in (1.1). It relies on a new developed rescaling method combined with a key doubling property. See [**PQS**] for the details. This new technique provides some new connections of the Liouville-type theorem in $\mathbb{R}^N$ and local properties of solutions.

Lemma 4. *(Doubling lemma) Let* (X, d) *be complete metric space and* $\emptyset \neq D \subset \Sigma \subset X$*, with* Σ *closed. Set* $\Gamma = \Sigma \setminus D$*. Let* $M : D \to (0, \infty)$ *be bounded on compact subsets of* D *and fix a positive number* n*. If* $y \in D$ *is such that*

$$M(y) dist(y, \Gamma) > 2n,$$

then there exists $x \in D$ *such that*

$$M(x) dist(x, \Gamma) > 2n, \qquad M(x) \geq M(y)$$

and

$$M(z) \leq 2M(x), \qquad \forall z \in D \cap \bar{B}(x, nM^{-1}(x)).$$

We next show the precise characterization of interior estimate of every solution by distance function. Adapting the spirits in [**PQS**] and doubling lemma, we have

Lemma 5. *(interior gradient estimate) Let* Ω *be arbitrary domain, then there exists* C *independent of* Ω *and* u *such that for any solution in* (1.1) *in* Ω*, there holds*

$$u + |Du|^{\frac{p}{q+1}} \leq C(1 + dist^{-\frac{p}{q+1-p}}(x, \partial\Omega)), \quad x \in \Omega. \tag{3.2}$$

Proof. Suppose that the estimate (3.2) is not true, then there exist sequences of $\Omega_n, u_n, y_n \in \Omega_n$ such that u_n solves (1.1) in Ω_n and the function

$$M_n := u_n^{\frac{q+1-p}{p}} + |Du_n|^{\frac{q+1-p}{q+1}}$$

satisfies

$$M_n(y_n) > 2n(1 + dist^{-1}(y_n, \partial\Omega_n) > 2n dist^{-1}(y_n, \partial\Omega_n).$$

Due to doubling lemma above, it follows that there exists $x_n \in \Omega_n$ such that

$$M_n(x_n) > 2n dist^{-1}(x_n, \partial\Omega_n)$$

and

$$M_n(z) \leq 2M_n(x_n) \ \text{ for } \ |z - x_n| \leq nM_n^{-1}(x_n).$$

Introduce a new function

$$v_n(y) := \frac{u_n(x_n + M_n^{-1} y)}{M_n^{\frac{p}{q+1-p}}(x_n)} \ \text{ in } \ |y| \leq n.$$

Since u_n satisfies

$$-\text{div}(1 + |Du_n(x)|^2)^{\frac{p}{2}-1} Du_n(x) = u_n^q(x), \qquad x \in \Omega,$$

we deduce that the function $v_n(y)$ satisfies

$$\int_{|y| \leq n} \Big(\frac{1}{M_n^{\frac{2(q+1)}{q+1-p}}(x_n)} + |Dv_n|^2\Big)^{\frac{p}{2}-1} Dv_n D\phi \, dy = \int_{|y| \leq n} v_n^q \phi \, dy \tag{3.3}$$

for any $\phi \in C_0^\infty(|y| \le n)$. Moreover,

$$[v_n^{\frac{q+1-p}{p}}(0) + |Dv_n|^{\frac{q+1-p}{q+1}}(0)] = 1 \tag{3.4}$$

and

$$[v_n^{\frac{q+1-p}{p}}(y) + |Dv_n|^{\frac{q+1-p}{q+1}}(y)] \le 2, \quad \forall\ |y| \le n.$$

For any smooth compact set D in $\mathbb{R}^N$, $|v_n(y)| \le 2^{\frac{p}{q+1-p}}$ in D if n is large enough. By $C^{1,\alpha}$ estimate in [**T**], we have

$$\|v_n\|_{C^{1,\alpha}(\bar{D})} \le C,$$

where C is a constant independent of v_n and n. Through Arzelá-Ascoli Theorem, there exists $v \in C^1(D)$ such that v_n converges to v in $C^1(D)$. Argue as Theorem 1, we obtain that v_n converges locally in $C^1(\mathbb{R}^N)$ to a solution $v \in C^1(\mathbb{R}^N)$ satisfying

$$\begin{cases} \int_{\mathbb{R}^N} |Dv|^{p-2} Dv D\phi \, dy = \int_{\mathbb{R}^N} v^q \phi \, dy, \\ v \ge 0, \end{cases}$$

as $n \to \infty$. Due to the Liouville-type theorem in lemma 1, v is trivial. However, (3.4) implies that $v(x) \not\equiv 0$. A contradiction is achieved. Thus, the lemma is completed. □

Employing the standard barrier method, we obtain the uniform boundary gradient estimate. See [**D**] for a similar argument.

LEMMA 6. *(Boundary gradient estimate) Assume that the mean curvature of $\partial\Omega$ has positive bounds H_0 from below and $1 < p < N$, then for every solution of (1.1), there exists some constant C independent of u such that* $\sup_{\partial\Omega} |Du| \le C$.

PROOF. Let $d(x)$ denote the distance function from the boundary, i.e. $d(x) = dist(x, \partial\Omega)$. From the appendix of Chapter 14 in [**GT**], $d(x)$ has the following properties:

(I) There exists positive ϱ such that $d(x) \in C^2(U)$ and $d(x)$ satisfy

$$\Delta d(x) \le -(N-1)H_0,$$

where $U := \Omega \cap \{x | d(x) < \varrho\}$.

(Π) We have $|Dd(x)| = 1$, then $\sum_{i=1}^{N} D_i d(x) D_{ij} d(x) = 0, \quad j = 1 \cdots N, \quad \forall x \in U.$

Introduce a new function $\theta(x) := md(x)$, where m is some positive constant that will be determined later. Deduced from the left item of (1.1) and (I-Π), we have

$$\begin{aligned} -\mathrm{div}[(1+|D\theta(x)|^2)^{\frac{p}{2}-1}D\theta(x)] &= \sum_{i,j=1}^{N} -[\frac{\delta_{ij}}{(1+|D\theta|^2)^{1-\frac{p}{2}}} + \frac{(p-2)D_i\theta D_j\theta}{(1+|D\theta|^2)^{2-\frac{p}{2}}}]D_{ij}\theta \\ &\geq \frac{m\Delta d(x)}{(1+m^2)^{1-\frac{p}{2}}} \\ &\geq (N-1)m^{p-1}H_0, \qquad \forall x \in U. \end{aligned}$$

Thanks to Theorem 1, we have $\|u\|_{L^\infty} \leq M$, which is independent of u. Since $p > 1$, there must exist m_0 such that for all $m > m_0$,

$$-\mathrm{div}[(1+|D\theta(x)|^2)^{\frac{p}{2}-1}D\theta(x)] \geq -\mathrm{div}[(1+|Du(x)|^2)^{\frac{p}{2}-1}Du(x)] = u^q, \quad x \in U \tag{3.5}$$

and

$$md(\varrho) \geq \|u\|_{L^\infty} = M. \tag{3.6}$$

Moreover,

$$\theta(x) = u(x) = 0, \qquad x \in \partial\Omega. \tag{3.7}$$

With (3.5), (3.6), (3.7) and the comparison principles in lemma 9. We infer that

$$0 \leq u(x) \leq \theta(x) = md(x), \qquad x \in U$$

for m large. Using the barrier method and the fact of $|Dd(x)| = 1$, there exists some constant C such that

$$\sup_{x\in\partial\Omega} |Du| \leq C.$$

Hence the lemma is verified. □

REMARK 2. *We would like to mention that our assumptions for the mean curvature of the boundary of Ω is necessary to some extent for this type of quasi-linear operator. In the case of mean curvature operator, i.e. $p = 1$ in our generalized mean curvature operator, the mean curvature of $\partial\Omega$ is required to be at least nonnegative at every point of $\partial\Omega$ in order to get the uniform gradient boundary estimate. See the chapter 14 in* [**GT**] *for more details. Also obviously our type of domain is achievable.*

PROOF OF COROLLARY 1. Using the interior gradient estimate and boundary gradient estimate above, then applying an argument similar to the proof of Theorem 8.29 in [**GT**] , we conclude

$$\sup_{x\in\bar{\Omega}} |Du(x)| \leq C.$$

From the structure condition (2.1), generalized mean curvature equation is strictly elliptic. Following from Theorem 6.2 of chapter 4 in [**LN**], u has second generalized derivatives. Furthermore, by Theorem 6.3 of chapter 4 in [**LN**], we have $\|u\|_{C^2(\bar{\Omega})} \leq C$. □

4. Existence

In this section we prove the existence of positive solution of (1.1). In order to achieve that, we need the following lemma about fixed point theorem [**DLN**](see also e.g. [**R**] and [**Z**] for the argument of p Laplacian equation) of compact operators in a cone.

LEMMA 7. *Let $\Re$ be a cone in a Banach space and $K : \Re \to \Re$ a compact operator such that $K(0) = 0$. Assume that there exists $r > 0$ such that:*
(a) $u \neq tK(u)$ for all $\|u\| = r$, $t \in [0, 1]$.

Assume also that there exist a compact homotopy $H : [0, 1] \times \Re \to \Re$, and $R > r$ such that:
(b1) $K(u) = H(0, u)$ for all $u \in \Re$.
(b2) $H(t, u) \neq u$ for any $\|u\| = R$ and $t \in [0, 1]$.
(b3) $H(1, u) \neq u$ for any $\|u\| \leq R$.

Let $D = \{u \in \Re : r < \|u\| < R\}$. Then, K has a fixed point in D.

To prove that existence, we consider a priori bounds of solutions of generalized mean curvature equation with parameter λ. We will need the non-existence of solutions for λ large. Assume that the nonnegative solutions solve

$$\begin{cases} -\text{div}((1 + |Du(x)|^2)^{\frac{p}{2}-1} Du(x)) = u^q(x) + \lambda, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda \geq 0$ is constant. We have the following a priori estimates. For more general equation, similar blow-up technique is carried out in [**ACM**].

LEMMA 8. *Let Ω be bounded smooth domain and $p - 1 < q < p^* - 1$. $u(x) \in W_0^{1,p}(\Omega)$ is any solution of (4.1), then there exists C independent of u and λ such that*

$$\|u\|_{L^\infty} + \lambda \leq C.$$

PROOF. We argue by contradiction and suppose that above conclusion is false. Then, there exists a sequence of solutions $\{u_n(x), \lambda_n\}$ such that

$$\lim_{n\to\infty} (\|u_n\|_{L^\infty(\Omega)} + \lambda_n) = \infty.$$

Assume that $M_n := u_n(x_n) = \sup_\Omega u_n(x)$. Next we consider two cases.

Case I: There holds (for a subsequence, still denoted by n)

$$\lim_{n\to\infty} \frac{\lambda_n}{M_n^q} = 0,$$

which implies that $M_n \to \infty$ as $n \to \infty$. Introduce

$$w_n(y) := \frac{u_n(M_n^{-k} y + x_n)}{M_n}$$

in Ω_n, where $\Omega_n := M_n^k(\Omega \setminus x_n)$ and $k := \frac{q-p+1}{p}$. Since $u_n(x)$ satisfies

$$\int_\Omega (1+|Du_n|^2)^{\frac{p}{2}-1} Du_n D\varphi\, dx = \int_\Omega (u_n^q + \lambda_n)\varphi\, dx$$

for any $\varphi \in C_0^\infty(\Omega)$, then w_n solves

$$\begin{cases} \int_{\Omega_n} (\frac{1}{M_n^{2(k+1)}} + |Dw_n|^2)^{\frac{p}{2}-1} Dw_n D\phi\, dy & = \int_{\Omega_n} (w_n^q + \frac{\lambda_n}{M_n^q})\phi\, dy, \\ \|w_n\|_{L^\infty(\Omega_n)} & \leq 1, \\ w_n(0) & = 1, \\ w_n(y) & = 0 \qquad \text{on } \partial\Omega_n \end{cases} \tag{4.2}$$

for all $\phi \in C_0^\infty(\Omega_n)$, where $\phi(y) := \varphi(M_n^{-k}y + x_n)$. As before, let $d_n := dist(x_n, \partial\Omega)$. Two subcases may occur. Either $M_n^k d_n \to \infty$ or $M_n^k d_n \to d$ for some $d > 0$ as $n \to \infty$. Argue exactly as in the proof of Theorem 1, we will arrive at a contradiction from the Liouville-type theorem of p laplacian equation in Euclidean space and half space since $\lim_{n\to\infty} \frac{\lambda_n}{M_n^q} = 0$ in the right hand side .

Case II: There exists $C_0 > 0$ such that

$$\liminf_{n\to\infty} \frac{\lambda_n^{\frac{1}{q}}}{M_n} \geq C_0,$$

which implies that $\lambda_n \to \infty$ as $n \to \infty$. For any $x_0 \in \Omega$, let

$$w_n(y) := \frac{u_n(\lambda_n^{-\frac{k}{q}} y + x_0)}{\lambda_n^{\frac{1}{q}}}$$

in Ω_n and $\Omega_n := \lambda_n^{\frac{k}{q}}(\Omega \setminus x_0)$. Then, $w_n(y)$ satisfies

$$\int_{\Omega_n} (\frac{1}{\lambda_n^{\frac{2(k+1)}{q}}} + |Dw_n|^2)^{\frac{p}{2}-1} Dw_n D\phi\, dy = \int_{\Omega_n} (w_n^q + 1)\phi\, dy \tag{4.3}$$

for any $\phi \in C_0^\infty(\Omega_n)$. Moreover

$$0 < w_n(y) \leq \frac{M_n}{\lambda_n^{\frac{1}{q}}} \leq \frac{1}{C_0}$$

in Ω_n. Since $dist(x_0, \partial\Omega)\lambda_n^{\frac{k}{q}} \to \infty$ as $n \to \infty$, then Ω_n converges to the entire space $\mathbb{R}^N$. From the same spirit in Theorem 1, using the Arzelá-Ascoli Theorem and diagonal line argument, there exists $w \in C^1(\mathbb{R}^N)$ such that $w_n \to w$ uniformly on any compact subset of $\mathbb{R}^N$ and w satisfies

$$\begin{cases} \int_{\mathbb{R}^N} |Dw|^{p-2} Dw D\phi\, dy = \int_{\mathbb{R}^N} (w^q + 1)\phi\, dy, \\ 0 \leq w(x) \leq \frac{1}{C_0}, \qquad \forall x \in \mathbb{R}^N \end{cases}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$. This contradicts lemma 3.

In conclusion, the lemma is completed. □

Since C in the above lemma is independent of λ, we have

COROLLARY 2. *There exists λ_0 such that problem (4.1) has no positive solution for any $\lambda > \lambda_0$.*

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let $C(\bar{\Omega})$, $C^1(\bar{\Omega})$, and $C^{1,\tau}(\bar{\Omega})$ be Banach spaces equipped with the standard norm. For each function $v \in C(\bar{\Omega})$, let $T(v) \in C^{1,\tau}(\bar{\Omega})$ $(\tau > 0)$ be the unique weak solution of the problem:

$$-\mathrm{div}((1 + |Tv(x)|^2)^{\frac{p}{2}-1} DTv(x)) = v(x), \quad x \in \Omega$$

with $Tv(x) = 0$ on $\partial\Omega$. It is true that $T : C(\bar{\Omega}) \to C^{1,\tau}(\bar{\Omega})$ is continuous operator. Let $F(u) := u^q$. Consider the operator

$$K := T \circ F : C^1(\bar{\Omega}) \to C^1(\bar{\Omega}).$$

From the compactness of the inclusion $C^{1,\tau}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$, the operator K is compact. Let $\Re := \{u \in C^1(\bar{\Omega}) | u \geq 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}$. Then, $\Re$ is a cone in $C^1(\bar{\Omega})$ and K maps $\Re$ into $\Re$ because of the comparison principle. We are going to find at least a nontrivial fixed point of K in $\Re$.

First, we verify condition (a). If $u \in \Re \setminus \{0\}$ is a solution of $u = tK(u)$ for some $t \in [0, 1]$, then u is a solution of the equation

$$-\mathrm{div}((t^2 + |Du(x)|^2)^{\frac{p}{2}-1} Du(x)) = t^{p-1} u^q(x), \quad x \in \Omega$$

with $u(x) = 0$ on $\partial\Omega$. Since $p \geq 2$, multiplying by u and integrating by part, we obtain

$$\begin{aligned}
\int_\Omega |Du|^p\,dx \leq \int_\Omega (t^2 + |Du|^2)^{\frac{p}{2}-1} |Du|^2\,dx &= t^{p-1} \int_\Omega u^{q+1}\,dx \\
&\leq \int_\Omega u^{q+1}\,dx \\
&\leq C\Big(\int_\Omega |Du|^p\,dx\Big)^{\frac{q+1}{p}}.
\end{aligned}$$

In above we have applied Hölder and Sobolev inequalities in last inequality. Since $\frac{q+1}{p} > 1$, then there exists some positive constant C such that $\int_\Omega |Du|^p\,dx > C$. Hence, we can choose $r > 0$ small enough such that for any $t \in [0, 1]$ the equation $u = tK(u)$ has no nontrivial solution in $\|u\|_{C^1(\bar{\Omega})} \leq r$. This implies that condition (a) in lemma 7 holds.

Define $H : [0, 1] \times \Re \to \Re$ as $H(t, u) = T[F(u) + t\lambda_0]$, where λ_0 is given in Corollary 2. Clearly condition $(b1)$ is true. Observe that $H(t, u) = u$ is actually equivalent to

$$\mathrm{div}((1 + |Du(x)|^2)^{\frac{p}{2}-1} Du(x)) + u^q(x) + t\lambda_0 = 0, \quad x \in \Omega \tag{4.4}$$

with $u(x) = 0$ on $\partial\Omega$. As in Theorem 1, we can prove that every solution in (4.4) has a priori bounds. Moreover, the C^1 norms is bounded. Then we can find some $R > 0$ such that every solution verifies $\|u\|_{C^1(\Omega)} < R$. Therefore, condition $(b2)$ holds. The condition $(b3)$ easily follows from Corollary 2. In conclusion, Theorem 2 is derived from lemma 7.

□

5. Non-existence

In this section, we study the non-existence of generalized mean curvature equation in bounded domain. Using the well-known Pohozaev identities (e.g. [**M**],[**P**]), we establish that there is no solution in star-shaped domain when $q \geq p^* - 1$, which is also the extension of laplacian equation.

Proof of Theorem 3. Since $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, from Theorem 5.1 of chapter 4 in [**LN**] and non-degenerate structure (2.1), u has second generalized derivatives. Moreover, due to Theorem 6.3 of chapter 4 in [**LN**], $u \in C^2(\bar{\Omega})$. We multiply the right hand side of (1.1) by $x \cdot Du$, then integrate by part,

$$\int_\Omega u^q (x \cdot Du)\, dx = \frac{1}{q+1} \int_\Omega x \cdot Du^{q+1}\, dx = -\frac{n}{q+1} \int_\Omega u^{q+1}\, dx. \tag{5.1}$$

Let $A := (1 + |Du|^2)$ for conveniences. Multiplying the left hand side of (1.1) by $x \cdot Du$, then integrating by part several times,

$$\begin{aligned}
\int_\Omega -\mathrm{div}(A^{\frac{p}{2}-1} Du)(x \cdot Du)\, dx &= \int_\Omega A^{\frac{p}{2}-1} |Du|^2\, dx + \sum_{i,j=1}^N \int_\Omega A^{\frac{p}{2}-1} x_j D_{ij} u D_i u\, dx \\
&\quad - \int_{\partial\Omega} A^{\frac{p}{2}-1} \frac{\partial u}{\partial \nu} (x \cdot Du)\, ds \\
&= \int_\Omega A^{\frac{p}{2}-1} |Du|^2\, dx + \frac{1}{p} \sum_{j=1}^N \int_\Omega D_j A^{\frac{p}{2}} \cdot x_j\, dx \\
&\quad - \int_{\partial\Omega} A^{\frac{p}{2}-1} \frac{\partial u}{\partial \nu} (x \cdot Du)\, ds \\
&= \int_\Omega A^{\frac{p}{2}-1} |Du|^2\, dx - \frac{n}{p} \int_\Omega A^{\frac{p}{2}}\, dx \\
&\quad + \frac{1}{p} \int_{\partial\Omega} A^{\frac{p}{2}} (x \cdot \nu)\, ds - \int_{\partial\Omega} A^{\frac{p}{2}-1} \frac{\partial u}{\partial \nu} (x \cdot Du)\, ds,
\end{aligned} \tag{5.2}$$

where ν is exterior unit normal. Since $u = 0$ on the $\partial\Omega$, then $\nu = -\frac{Du}{|Du|}$. Hence,

$$\frac{\partial u}{\partial \nu} (x \cdot Du) = |Du|^2 (x \cdot \nu). \tag{5.3}$$

Together with (1.1), (5.1), (5.2) and (5.3), we have

$$\frac{n}{p}\int_\Omega A^{\frac{p}{2}}\,dx-\int_\Omega A^{\frac{p}{2}-1}|Du|^2\,dx-\frac{n}{q+1}\int_\Omega u^{q+1}\,dx = \frac{1}{p}\int_{\partial\Omega}A^{\frac{p}{2}}(x\cdot\nu)\,ds - \int_{\partial\Omega}A^{\frac{p}{2}-1}(x\cdot\nu)\,ds. \tag{5.4}$$

Note that $A^{\frac{p}{2}}=A^{\frac{p}{2}-1}(1+|Du|^2)$. We rewrite (5.4) in the form of

$$\frac{n}{p}\int_\Omega A^{\frac{p}{2}-1}\,dx+\frac{n-p}{p}\int_\Omega A^{\frac{p}{2}-1}|Du|^2\,dx-\frac{n}{q+1}\int_\Omega u^{q+1}\,dx = \frac{1-p}{p}\int_{\partial\Omega}A^{\frac{p}{2}-1}|Du|^2(x\cdot\nu)\,ds - \int_{\partial\Omega}A^{\frac{p}{2}-1}(x\cdot\nu)\,ds. \tag{5.5}$$

Multiplying both side of (1.1) by u, then integrating by part, we get

$$\int_\Omega A^{\frac{p}{2}-1}|Du|^2\,dx=\int_\Omega u^{q+1}\,dx. \tag{5.6}$$

Then, together with(5.6), (5.5) could be written as

$$\frac{n}{p}\int_\Omega A^{\frac{p}{2}-1}\,dx+\left(\frac{n-p}{p}-\frac{n}{q+1}\right)\int_\Omega u^{q+1} = \frac{1-p}{p}\int_{\partial\Omega}A^{\frac{p}{2}-1}|Du|^2(x\cdot\nu)\,ds - \int_{\partial\Omega}A^{\frac{p}{2}-1}(x\cdot\nu)\,ds. \tag{5.7}$$

Since Ω is strictly star-shaped with respect to $0\in\mathbb{R}^N$, we have $x\cdot\nu>0$ for all $x\in\partial\Omega$. Moreover, with $p>1$,

$$\frac{1-p}{p}\int_{\partial\Omega}A^{\frac{p}{2}-1}|Du|^2(x\cdot\nu)\,ds-\int_{\partial\Omega}A^{\frac{p}{2}-1}(x\cdot\nu)\,ds<0.$$

While

$$\frac{n}{p}\int_\Omega A^{\frac{p}{2}-1}\,dx+\left(\frac{n-p}{p}-\frac{n}{q+1}\right)\int_\Omega u^{q+1}>0$$

in the case of $\frac{n-p}{p}-\frac{n}{q+1}\geq 0$, i.e. $q\geq p^*-1$. Clearly the above two inequalities contradict identity of (5.7). Therefore, the theorem is completed. □

REMARK 3. *The authors in* [**SY**] *consider the non-existence theorem of positive solutions of a general Euler-Lagrange equation which includes* (1.1). *They only show the non-existence for the case of* $p\geq 2$.

6. Appendix

Before showing the comparison principle, we study some vector calculations, which is useful later. Let a and b be vectors in $\mathbb{R}^N$. The formula

$$(1+|b|^2)^{\frac{p}{2}-1}b-(1+|a|^2)^{\frac{p}{2}-1}a=\int_0^1\frac{d}{dt}(1+|a+t(b-a)|^2)^{\frac{p}{2}-1}(a+t(b-a))\,dt$$

yields

$$(1+|b|^2)^{\frac{p}{2}-1}b-(1+|a|^2)^{\frac{p}{2}-1}a=(b-a)\int_0^1(1+|a+t(b-a)|^2)^{\frac{p}{2}-1}\,dt$$
$$+(p-2)\int_0^1(1+|a+t(b-a)|^2)^{\frac{p}{2}-2}\langle a+t(b-a),b-a\rangle(a+t(b-a))\,dt.$$

Then

$$\langle(1+|b|^2)^{\frac{p}{2}-1}b-(1+|a|^2)^{\frac{p}{2}-1}a,b-a\rangle=|b-a|^2\int_0^1(1+|a+t(b-a)|^2)^{\frac{p}{2}-1}\,dt$$
$$+(p-2)\int_0^1(1+|a+t(b-a)|^2)^{\frac{p}{2}-2}(\langle a+t(b-a),b-a\rangle)^2\,dt. \tag{6.1}$$

If $p\geq 2$, clearly

$$\langle(1+|b|^2)^{\frac{p}{2}-1}b-(1+|a|^2)^{\frac{p}{2}-1}a,b-a\rangle\geq 0. \tag{6.2}$$

If $1<p<2$, simple calculation shows

$$\langle(1+|b|^2)^{\frac{p}{2}-1}b-(1+|a|^2)^{\frac{p}{2}-1}a,b-a\rangle\geq$$
$$(p-1)|b-a|^2\int_0^1(1+|a+t(b-a)|^2)^{\frac{p}{2}-2}|a+t(b-a)|^2\,dt\geq 0. \tag{6.3}$$

In short,

$$\langle(1+|b|^2)^{\frac{p}{2}-1}b-(1+|a|^2)^{\frac{p}{2}-1}a,b-a\rangle>0 \tag{6.4}$$

for any $p>1$ if $b\neq a$.

Lemma 9. *(Comparison Principle) Suppose u_1, $u_2\in W^{1,p}(\Omega)$ satisfy*

$$\begin{cases}-\mathrm{div}((1+|Du_1|^2)^{\frac{p}{2}-1}Du_1)\leq-\mathrm{div}((1+|Du_2|^2)^{\frac{p}{2}-1}Du_2) & \text{in } \Omega,\\ u_1\leq u_2 & \text{on } \partial\Omega.\end{cases} \tag{6.5}$$

Then, $u_1\leq u_2$ in Ω.

Proof. Given $\epsilon>0$, the set

$$D_\epsilon:=\{x|u_1(x)>u_2(x)+\epsilon\}$$

is empty or $D_\epsilon\subset\subset\Omega$. Let

$$\eta(x):=\max\{u_1(x)-u_2(x)-\epsilon,\,0\},$$

then $\eta(x)\in W_0^{1,p}(\Omega)$. Multiplying (6.5) by $\eta(x)$, we get

$$\int_{D_\epsilon}\langle(1+|D_1u|^2)^{\frac{p}{2}-1}Du_1-(1+|D_2u|^2)^{\frac{p}{2}-1}Du_2,\,Du_1-Du_2\rangle\leq 0.$$

The above is possible only if $Du_1=Du_2$ a.e. in D_ϵ because of (6.4). Thus, $u_1(x)=u_2(x)+C$ in D_ϵ and $C=\epsilon$ since $u_1(x)=u_2(x)+\epsilon$ on ∂D_ϵ. Therefore, $u_1(x)\leq u_2(x)+\epsilon$ in Ω. It follows that $u_1(x)\leq u_2(x)$ in Ω as $\epsilon\to 0$. □

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