Fractional equations with indefinite nonlinearities

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Abstract

In this paper, we consider a fractional equation with indefinite nonlinearities

$$(-\triangle)^{\alpha/2}u = a(x_1)f(u)$$

for $0 < \alpha < 2$, where a and f are nondecreasing functions. We prove that there is no positive bounded solution. In particular, this remarkably improves the result in [CZ] by extending the range of α from $[1, 2)$ to $(0, 2)$, due to the introduction of new ideas, which may be applied to solve many other similar problems.

Key words: The fractional Laplacian, indefinite nonlinearities, method of moving planes, monotonicity, non-existence of positive solutions.

1 Introduction

The fractional Laplacian in R^n is a nonlocal pseudo-differential operator, assuming the form

$$\begin{aligned}(-\Delta)^{\alpha/2}u(x) &= C_{n,\alpha}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(z)}{|x-z|^{n+\alpha}}dz \\ &= C_{n,\alpha} \lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x)-u(z)}{|x-z|^{n+\alpha}}dz, \qquad (1)\end{aligned}$$

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where α is any real number between 0 and 2 and P.V. stands for Cauchy principal value. In order the integral to make sense, we require $u \in L_\alpha \cap C^{1,1}_{loc}$, where

$$L_\alpha = \{u \in L^1_{loc}(R^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha}}\, dx < \infty\}.$$

We will assume that u satisfies this condition through out the paper.

The non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [CS] introduced the *extension method* that reduced this nonlocal problem into a local one in higher dimensions. For a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, consider the extension $U : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ that satisfies

$$\begin{cases} div(y^{1-\alpha}\nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x). \end{cases}$$

Then

$$(-\triangle)^{\alpha/2} u(x) = -C_{n,\alpha} \lim_{y \longrightarrow 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \quad x \in \mathbb{R}^n.$$

This *extension method* has been applied successfully to study equations involving the fractional Laplacian, and a series of fruitful results have been obtained (see [BCPS] [CZ] and the references therein).

In [BCPS], among many interesting results, when the authors considered the properties of the positive solutions for

$$(-\triangle)^{\alpha/2} u = u^p(x), \quad x \in \mathbb{R}^n, \tag{2}$$

they first used the above *extension method* to reduce the nonlocal problem into a local one for $U(x, y)$ in one higher dimensional half space $\mathbb{R}^n \times [0, \infty)$, then applied the *method of moving planes* to show the symmetry of $U(x, y)$ in x, and hence derived the non-existence in the subcritical case:

Proposition 1 *(Brandle-Colorado-Pablo-Sanchez) Let $1 \leq \alpha < 2$. Then the problem*

$$\begin{cases} div(y^{1-\alpha}\nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ -\lim\limits_{y \longrightarrow 0^+} y^{1-\alpha} \dfrac{\partial U}{\partial y} = U^p(x, 0), & x \in \mathbb{R}^n \end{cases} \tag{3}$$

has no positive bounded solution provided $p < (n + \alpha)/(n - \alpha)$.

Then They took trace to obtain

Corollary 1 *Assume that* $1 \leq \alpha < 2$ *and* $1 < p < \frac{n+\alpha}{n-\alpha}$. *Then equation (2) possesses no bounded positive solution.*

A similar *extension method* was adapted in [CZ] to obtain the nonexistence of positive solutions for an indefinite fractional problem:

Proposition 2 *(Chen-Zhu) Let* $1 \leq \alpha < 2$ *and* $1 < p < \infty$. *Then the equation*

$$(-\triangle)^{\alpha/2} u = x_1 u^p, \quad x \in \mathbb{R}^n \tag{4}$$

possesses no positive bounded solution.

The common restriction $\alpha \geq 1$ is due to the approach that they need to carry the *method of moving planes* on the solutions U of the extended problem

$$div(y^{1-\alpha} \nabla U) = 0, \quad (x, y) \in \mathbb{R}^n \times [0, \infty). \tag{5}$$

Due to the presence of the factor $y^{1-\alpha}$, they have to assume that $\alpha \geq 1$, and it seems that this condition cannot be weakened if one wants to carry out the *method of moving planes* on extended equation (5). However, this obstacle does not appear in equation (4), hence one may expect to be able to remove the condition if working on it directly.

In [CLL], when studying equation (2), the authors applied the *method of moving planes* directly to it without making an *extension* and thus obtain

Proposition 3 *Assume that* $0 < \alpha < 2$ *and* u *is a nonnegative solution of equation (2). Then*

(i) In the critical case $p = \frac{n+\alpha}{n-\alpha}$, u *is radially symmetric and monotone decreasing about some point.*

(ii) In the subcritical case $1 < p < \frac{n+\alpha}{n-\alpha}$, $u \equiv 0$.

This greatly improves the result in Corollary 1 by extending the range of α from $[1, 2)$ to $(0, 2)$.

In this paper, we will modify the *direct method of moving planes* introduced in [CLL], so that it can be applied to equation (4) here without going through extension. There are several difficulties.

Usually, to carry on the method of moving planes, one needs to assume that the solution u vanishes at ∞. For equation (2), without assuming $\lim_{|x| \to \infty} u(x) = 0$, in the critical and subcritical cases, one can exploit the Kelvin transform $v(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2})$ to derive

$$(-\triangle)^{\alpha/2} v(x) = \frac{1}{|x|^{\gamma}} v^p(x), \quad \text{with} \lim_{|x| \to \infty} v(x) = 0. \tag{6}$$

Here $\gamma \geq 0$ and the coefficient $\frac{1}{|x|^\gamma}$ possesses the needed monotonicity, so that one can carry on the method of moving planes on the transformed equation (6).

Now for equation (2), due to the presence of x_1, the coefficient of the transformed equation does not have the required monotonicity, and this renders the Kelvin transform useless.

To assume $\lim_{|x|\to\infty} u(x) = 0$ is impractical, because when in the process of applying this Liouville Theorem (nonexistence of solutions) in the blowing up and re-scaling arguments to establish a priori estimate, the solution of the limiting equation is known to be only bounded. Hence it is reasonable to assume that u is bounded when we consider equation (4). Without the condition $\lim_{|x|\to\infty} u(x) = 0$, in order to use the method of moving planes, we introduce an auxiliary function. As we will explain in the next section, the situation in the fractional order equation is quite different and more difficult than the one in the integer order equation, and to overcome these difficulties, we introduce some new ideas when we move the planes along x_1 direction all the way up to ∞. Specifically, we consider a more general equation than (4),

$$(-\triangle)^{\alpha/2} u = a(x_1) f(u). \tag{7}$$

The conditions summarized as (H) on the function a and f are assumed as follows:
(H1): $a(t) \in C^s(\mathbb{R})$ for some $s \in (0,1)$ and $a(t)$ is nondecreasing in $\mathbb{R}$.
(H2): $a(t) \leq 0$ for $t \leq 0$ or $a(t) = o(|t|^{-\alpha})$ for $t \to -\infty$ and $a(t) > 0$ somewhere for $t > 0$.
(H3): f is locally Lipschitz and nondecreasing in $(0, \infty)$. Moreover, $f(0) = 0$ and $f > 0$ in $(0, \infty)$.
We provide a basic and self-contained argument to derive the following theorem.

Theorem 1 *Let $0 < \alpha < 2$. Suppose u is a positive bounded solution of equation (7), then u is monotone increasing in x_1 direction.*

Upon the completion of the work, we noticed that the Liouville type theorem for indefinite fractional problem was considered in [BDGQ]. The authors were able to show the nonexistence of solutions for (4) for the range of $0 < \alpha < 2$ by the method of moving planes involving Green functions. General assumptions as (H) were studied in [BDGQ] as well. However, our assumption for $a(t)$ seems to be more general. Let us remark on our general assumption on $a(t)$.

Remark 1 *I): The assumption $a(t) \leq 0$ for $t \leq 0$ is required in [BDGQ]. Based on our arguments, if $a(t)$ decays faster than $|t|^{-\alpha}$ as $t \to -\infty$, our arguments can still carry out. The interested readers may refer the crucial inequality (20) for the details. For example, the following case is included our assumption (H),*

$$a(t) = \begin{cases} \big((t-1)^-\big)^{-\alpha-\epsilon}, & t < 0, \\ \big((t+1)^+\big)^{\alpha_1}, & t \geq 0 \end{cases}$$

for some $\epsilon > 0$ and $\alpha_1 > 0$. However, it does not satisfy the assumptions in [BDGQ].
II): Instead of assuming $a(0) = 0$ and $a(t) > 0$ in [BDGQ], we only assume $a(t)$ is positive somewhere for $t > 0$. So the example $a(t) = \left((t-1)^+\right)^m$ for $m > 0$ only holds in our case.
III): The local version $\alpha = 2$ as (7) was considered in [BCN], [Lin], [DL], to just mention a few. Note that our assumptions on $a(t)$ are even more general than those in the literature.

By comparing the solution u with the first eigenfunction at a unit ball far away from the origin, we derive a contradiction and hence prove

Theorem 2 *Let $0 < \alpha < 2$ and $h(t) \to \infty$ as $t \to \infty$. Then equation (7) possesses no positive bounded solution.*

In Section 2, we used a *direct method of moving planes* to derive the monotonicity of solutions along x_1 direction and prove Theorem 1. In Section 3, we establish the nonexistence of positive solutions and obtain Theorem 2. The last section is the appendix which provides the proof for an elementary lemma.

For more related articles, please see [CFY], [CL], [CL1], [CLL1], [CLLg], [CLO], [CLO1], [FC], [HLZ], [JW], [LZ], [LZ1] and the references therein.

2 Monotonicity of solutions

Consider

$$(-\triangle)^{\alpha/2} u(x) = x_1 u^p(x), \quad x \in \mathbb{R}^n. \tag{8}$$

We will use the *direct method of moving planes* to show that every positive solution must be strictly monotone increasing along x_1 direction and thus prove Theorem 1.

Let

$$T_\lambda = \{x \in \mathbb{R}^n |\, x_1 = \lambda, \text{ for some } \lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda = \{x \in \mathbb{R}^n |\, x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of x about the plane T_λ.

Assume that u is a solution of pseudo differential equation (8). To compare the values of $u(x)$ with $u_\lambda(x) \equiv u(x^\lambda)$, we denote

$$w_\lambda(x) = u_\lambda(x) - u(x).$$

From the assumption (H1), it follows that

$$(-\triangle)^{\alpha/2} w_\lambda(x) = \big(a(x_1^\lambda) - a(x_1)\big) f(u_\lambda) + a(x_1)\big(f(u_\lambda) - f(u)\big) \geq a(x_1) f'(\xi_\lambda) w_\lambda(x), \tag{9}$$

where $\xi_\lambda(x)$ is valued between $u(x)$ and $u_\lambda(x)$.

We want to show that

$$w_\lambda \geq 0 \;\; \forall x \in \Sigma_\lambda \text{ and for all } \lambda \in (-\infty, \infty).$$

To this end, usually a contradiction argument is used. Suppose w_λ has a negative minimum in Σ_λ, then one would derive a contradiction with inequality (9). However, here we only assume that u is bounded, which cannot prevent the minimum of $w_\lambda(x)$ from leaking to ∞. To overcome this difficulty, for integer order equations (see [Lin])

$$-\triangle u = x_1 u^p(x) \; x \in R^n,$$

an auxiliary function was introduced:

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)} \;\text{ with } g(x) \to \infty, \text{ as } |x| \to \infty.$$

Now

$$\lim_{|x| \to \infty} \bar{w}_\lambda(x) = 0$$

and hence $\bar{w}_\lambda$ can attain its negative minimum in the interior of Σ_λ. The corresponding left hand side of (9) becomes

$$-\triangle w_\lambda = -\triangle \bar{w}_\lambda \cdot g - 2 \bigtriangledown \bar{w}_\lambda \cdot \bigtriangledown g - \bar{w}_\lambda \cdot \triangle g. \tag{10}$$

At a minimum of $\bar{w}_\lambda$, the middle term on the right hand side vanishes since $\bigtriangledown \bar{w}_\lambda = 0$. This makes the analysis easier. However, the fractional counter part of (10) is

$$(-\triangle)^{\alpha/2} w_\lambda = (-\triangle)^{\alpha/2} \bar{w}_\lambda \cdot g - 2C \int_{R^n} \frac{(\bar{w}_\lambda(x) - \bar{w}_\lambda(y))(g(x) - g(y))}{|x-y|^{n+\alpha}} dy + \bar{w}_\lambda \cdot (-\triangle)^{\alpha/2} g.$$

At a minimum of $\bar{w}_\lambda$, the middle term on the right hand side (the integral) neither vanish nor has a definite sign. This is the main difficulty encountered by the fractional nonlocal operator, and to circumvent which, we introduce a different auxiliary function and estimate $(-\triangle)^{\alpha/2} w_\lambda$ in an entirely new approach. We believe that this new idea may be applied to study other similar problems involving fractional operators.

Step 1.

As usual, in the first step of the method of moving planes, we show that, for λ close to negative infinity, we have

$$w_\lambda(x) \geq 0, \;\; \forall\, x \in \Sigma_\lambda. \tag{11}$$

To this end, one usually uses a contradiction argument. Suppose there is a negative minimum x^o of w_λ, then one would try to show that

$$(-\triangle)^{\alpha/2} w_\lambda(x^o) < a(x_1^o) f'(\xi_\lambda(x^o)) w_\lambda(x^o)$$

which is a contradiction to inequality (9). In order w_λ to possess such a negative minimum, one obvious condition to impose on it is $\lim_{|x|\to\infty} w_\lambda(x) = 0$, which is almost equivalent to $\lim_{|x|\to\infty} u(x) = 0$. However, this condition is too strong in practice. The non-existence of solutions of (8) is used as an important ingredient in obtaining a priori estimate on the solutions by applying a blowing-up and re-scaling argument, and the solutions of the limiting equations are bounded, but may not goes to zero at infinity. Hence it is more reasonable to assume that the solutions are bounded, hence w_λ is bounded.

Different from the logarithmic auxiliary function chosen in [Lin] and [CZ], we choose the auxiliary function as

$$g(x) = |x - Re_1|^\sigma, \;\; \bar{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)},$$

where

$$R = \lambda + 1, \; e_1 = (1, 0, \cdots, 0),$$

and σ is a small positive number to be chosen later.

Obviously, $\bar{w}_\lambda$ and w_λ have the same sign and

$$\lim_{|x|\to\infty} \bar{w}_\lambda(x) = 0.$$

Now suppose (11) is violated, then there exists a negative minimum x^o of $\bar{w}_\lambda$, at which we compute:

$$
\begin{aligned}
& (-\triangle)^{\alpha/2} w_\lambda(x^o) = C\,PV \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+\alpha}} dy + C \int_{R^n \setminus \Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+\alpha}} dy \\
= \;& C\,PV \int_{\Sigma_\lambda} \left\{ \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+\alpha}} + \frac{w_\lambda(x^o) + w_\lambda(y)}{|x^o - y^\lambda|^{n+\alpha}} \right\} dy \\
= \;& C\,PV \int_{\Sigma_\lambda} [\bar{w}_\lambda(x^o) - \bar{w}_\lambda(y)] g(y) \left(\frac{1}{|x^o - y|^{n+\alpha}} - \frac{1}{|x^o - y^\lambda|^{n+\alpha}} \right) dy \\
+ \;& C \bar{w}_\lambda(x^o) \left\{ \int_{\Sigma_\lambda} \frac{2g(x^o)}{|x^o - y^\lambda|^{n+\alpha}} dy + PV \int_{\Sigma_\lambda} [g(x^o) - g(y)] \left(\frac{1}{|x^o - y|^{n+\alpha}} - \frac{1}{|x^o - y^\lambda|^{n+\alpha}} \right) dy \right\} \\
\leq \;& C \bar{w}_\lambda(x^o) \left\{ \int_{\Sigma_\lambda} \frac{2g(x^o)}{|x^o - y^\lambda|^{n+\alpha}} dy + PV \int_{\Sigma_\lambda} [g(x^o) - g(y)] \left(\frac{1}{|x^o - y|^{n+\alpha}} - \frac{1}{|x^o - y^\lambda|^{n+\alpha}} \right) dy \right\} \\
\leq \;& C \bar{w}_\lambda(x^o) \left\{ \int_{\Sigma_\lambda} \frac{g(x^o)}{|x^o - y^\lambda|^{n+\alpha}} dy + PV \int_{\Sigma_\lambda} \left[\frac{g(x^o) - g(y)}{|x^o - y|^{n+\alpha}} + \frac{g(y)}{|x^o - y^\lambda|^{n+\alpha}} \right] dy \right\} \\
\equiv \;& C \bar{w}_\lambda(x^o)(I_1 + I_2). \qquad (12)
\end{aligned}
$$

Here we have used the anti-symmetry property $w_\lambda(y^\lambda) = -w_\lambda(y)$.

By elementary calculus,

$$
g(x^o) \int_{|x_1^o - \lambda|}^{\infty} \frac{|S_r|}{r^{n+\alpha}} dr \geq I_1 \geq g(x^o) \frac{1}{4} \int_{\sqrt{2}|x_1^o - \lambda|}^{\infty} \frac{|S_r|}{r^{n+\alpha}} dr = \frac{c_1 g(x^o)}{|x_1^o - \lambda|^\alpha} \qquad (13)
$$

with a positive constant c_1.

We split I_2 into three parts:

$$
\begin{aligned}
I_2 \;&=\; (-\triangle)^{\alpha/2} g(x^o) - \int_{\Sigma_\lambda} \frac{g(x^o) - g(y)}{|x^o - y^\lambda|^{n+\alpha}} dy + \int_{\Sigma_\lambda} \frac{g(y^\lambda)}{|x^o - y^\lambda|^{n+\alpha}} dy \\
&\geq\; (-\triangle)^{\alpha/2} g(x^o) - I_{21}, \qquad (14)
\end{aligned}
$$

where

$$
I_{21} = \int_{\Sigma_\lambda} \frac{g(x^o) - g(y)}{|x^o - y^\lambda|^{n+\alpha}} dy.
$$

To calculate $(-\triangle)^{\alpha/2} g(x^o)$, we simply employ the following

Lemma 2.1 *Assume that $\gamma < \alpha$.*

$$
(-\triangle)^{\alpha/2}(|x - a|^\gamma) = C_\gamma |x - a|^{\gamma - \alpha}, \qquad (15)
$$

where C_γ is a constant continuously depending on γ,

$$C_\gamma : \begin{cases} > 0, & \text{if } \alpha - n < \gamma < 0; \\ = 0, & \text{if } \gamma = 0 \text{ or } \alpha - n; \\ < 0, & \text{if } 0 < \gamma < \alpha. \end{cases}$$

The proof is elementary, and will be given in the Appendix.

From this Lemma, we have

$$(-\triangle)^{\alpha/2} g(x^o) = \frac{C_\sigma}{|x^o - Re_1|^{\alpha-\sigma}} = \frac{C_\sigma g(x^o)}{|x^o - Re_1|^{\alpha}}, \tag{16}$$

where C_σ can be made as small as we wish for sufficiently small σ.

Evaluate the integral in I_{21} in two regions

$$D_1 = \Sigma_\lambda \cap (|y| \leq K|x^o|) \text{ and } D_2 = \Sigma_\lambda \cap (|y| > K|x^o|).$$

In D_1, due to our choice of $R = \lambda + 1$,

$$\begin{aligned} |g(x^o) - g(y)| &\leq |\triangledown g(\xi)||x^o - y| \\ &= \frac{\sigma}{|\xi - Re_1|^{1-\sigma}} |x^o - y| \\ &\leq C_2 |\triangledown g(x^o)||x^o - y| \\ &\leq C_2 \sigma g(x^o) \end{aligned}$$

where $\xi \in \Sigma_\lambda$ is some point on the line segment from x^o to y and C_2 is some positive constant depending only on x^o. Consequently,

$$\begin{aligned} & |\int_{D_1} \frac{g(x^o) - g(y)}{|x^o - y^\lambda|^{n+\alpha}} dy| \\ \leq\ & C_2 \sigma g(x^o) |\int_{D_1} \frac{1}{|x^o - y^\lambda|^{n+\alpha}} dy| \\ \leq\ & \frac{C_2 \sigma g(x^o)}{|x_1^o - \lambda|^\alpha}. \end{aligned} \tag{17}$$

On D_2, notice that

$$|g(x^o) - g(y)| \leq |g(x^o)| + |g(y)| \leq C_3 |y|^\sigma \text{ and } |x^o - y^\lambda| \geq |x^o - y| \sim |y|,$$

we derive

$$|\int_{D_2}\frac{g(x^o)-g(y)}{|x^o-y^\lambda|^{n+\alpha}}dy|\leq C_3\int_{D_2}\frac{|y|^\sigma}{|y|^{n+\alpha}}dy|\leq\frac{C_3}{(K|x^o|)^{\alpha-\sigma}}. \tag{18}$$

Combining (17) and (18), we arrive at

$$|I_{21}|\leq(C_2\sigma+\frac{C_3}{K^{\alpha-\sigma}})\frac{g(x^o)}{|x_1^o-\lambda|^\alpha}. \tag{19}$$

Taking into account of (12), (13), (14), and (16), we obtain

$$\begin{aligned}(-\triangle)^{\alpha/2}w_\lambda(x^o) &\leq C\bar{w}_\lambda(x^o)(I_1+I_2)\\ &\leq C\bar{w}_\lambda(x^o)\left(C_1+C_\sigma-C_2\sigma-\frac{C_3}{K^{\alpha-\sigma}}\right)\frac{g(x^o)}{|x_1^o-\lambda|^\alpha}\\ &\leq Cw_\lambda(x^o)\frac{1}{|x_1^o-\lambda|^\alpha}. \end{aligned}\tag{20}$$

To derive the last inequality above, we choose K large, then let σ be sufficiently small (hence C_σ becomes sufficiently small), which implies $I_1+I_2>0$ in the mean time. If $a(x_1)\leq 0$ for $x_1<0$, it is obvious that

$$(-\triangle)^{\alpha/2}w_\lambda(x^o)<a(x_1^o)f'(\xi_\lambda)w_\lambda(x^o),$$

which contradicts (9). If $a(x_1)=o(|x_1|^{-\alpha})$ as $x_1\to-\infty$, since $|x_1^0-\lambda|^{-\alpha}>a(x_1)f'(\xi_\lambda)$ as λ close to negative infinity, then

$$Cw_\lambda(x^o)\frac{1}{|x_1^o-\lambda|^\alpha}<a(x_1^o)f'(\xi_\lambda)w_\lambda(x^o).$$

It also provides a contradiction with (9).

Now we have completed *Step 1.* That is, we have shown that for all λ close to negative infinity, it holds

$$w_\lambda(x)\geq 0,\quad \forall\, x\in\Sigma_\lambda.$$

Step 2.

The above inequality provides a starting point to move the plane. Now we move plane T_λ towards the right as long as the inequality holds. We will show that T_λ can be moved all the way to infinity. More precisely, let

$$\lambda_o=\sup\{\lambda\mid w_\mu(x)\geq 0,\ x\in\Sigma_\mu,\ \mu\leq\lambda\},$$

and we will show that $\lambda_o=\infty$.

Suppose the contrary, $\lambda_o < \infty$. Then by the definition of λ_o, there exists a sequence of numbers $\{\lambda_k\}$, with $\lambda_k \searrow \lambda_o$, and $x^k \in \Sigma_{\lambda_k}$, such that

$$w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} < 0. \tag{21}$$

By (9) and (20), there exists a constant $c_o > 0$, such that

$$\frac{c_o}{|x_1^k - \lambda_k|^\alpha} \leq a(x_1^k) f'(\xi_\lambda(x^k)). \tag{22}$$

Denote

$$x = (x_1, y), \ \ x^k = (x_1^k, y^k) \quad \text{and} \quad u_k(x) = u(x_1, y - y^k).$$

Then $u_k(x)$ satisfies the same equation as $u(x)$ does. Notice that both u_k and $(-\triangle)^{\alpha/2} u_k$ are bounded, by Sobolev embeddings and regularity arguments, for some $\epsilon_1 > 0$, one can derive a uniform $C^{\alpha+\epsilon_1}$ estimate on $\{u_k\}$ (for example, see [CLL1]), and hence concludes that there is a nonnegative ($\not\equiv 0$) function u_0, such that

$$u_k(x) {\rightarrow} u_0(x) \quad \text{and} \quad (-\triangle)^{\alpha/2} u_0 = a(x_1) f(u_0), \ \ x \in \mathbb{R}^n.$$

Let

$$w_{k,\lambda}(x) = u_k(x^\lambda) - u_k(x) = u_{k,\lambda}(x) - u_k(x)$$

and

$$w_{0,\lambda}(x) = u_0(x^\lambda) - u_0(x) = u_{0,\lambda}(x) - u_0(x).$$

Then obviously

$$w_{k,\lambda}(x) {\rightarrow} w_{0,\lambda}(x), \quad \text{as } k {\rightarrow} \infty,$$

and due to the bounded-ness of $\{x_1^k\}$, there exists a subsequence (still denoted by $\{x_1^k\}$) which converges to x_1^0. Hence

$$w_{0,\lambda_o}(x_1^0, 0) = \lim w_{\lambda_k}(x^k) \leq 0.$$

On the other hand, for each $x \in \Sigma_{\lambda_o}$, we have

$$0 \leq w_{k,\lambda_o}(x) {\rightarrow} w_{0,\lambda_o}(x).$$

Furthermore, from the equation

$$(-\triangle)^{\alpha/2} w_{0,\lambda_o}(x) = a(x_1^{\lambda_0}) f(u_{0,\lambda_o}) - a(x_1) f(u_0), \quad x \in \Sigma_{\lambda_o}, \tag{23}$$

By the strong maximum principle, we must have

$$w_{0,\lambda_o}(x) > 0 \text{ for each } x \in \Sigma_{\lambda_o} \tag{24}$$

or

$$w_{0,\lambda_o}(x) \equiv 0 \text{ for each } x \in \Sigma_{\lambda_o}. \tag{25}$$

If (25) holds, then $a(x_1^{\lambda_0}) = a(x_1)$ in Σ_{λ_o}, which contradicts the assumption (H1) and (H2). Therefore, (24) must be true, which implies that

$$x_1^0 = \lambda_o.$$

It follows that

$$|x_1^k - \lambda_k| \to 0, \quad \text{as } k \to \infty.$$

This contradicts (22).

Hence, we must have

$$\lambda_o = \infty. \tag{26}$$

It follows that

$$w_\lambda(x) \geq 0, \ \forall\, x \in \Sigma_\lambda, \quad \text{for all } \lambda \in (-\infty, \infty).$$

Or equivalently, u is monotone increasing in x_1 direction.

This completes the proof of Theorem 1.

3 Non-existence of solutions

In the previous section, we have shown that positive solutions of the equation

$$(-\triangle)^{\alpha/2} u = a(x_1) f(u), \quad x \in \mathbb{R}^n \tag{27}$$

are monotone increasing along x_1 direction. Base on this, we will derive a contradiction, and hence prove the non-existence.

Proof of Theorem 2. Since $a(x_1)$ is positive somewhere and nondecreasing, we may assume that $a(x_1)$ is positive for $(R-2,\ \infty)$ for some large R. Let $\mathbb{B}_1(Re_1)$ be the unit ball centered at $(R, 0, \cdots, 0)$. Let ϕ be the first eigenfunction associated with $(-\triangle)^{\alpha/2}$ in $\mathbb{B}_1(Re_1)$:

$$\begin{cases} (-\triangle)^{\alpha/2} \phi(x) = \lambda_1 \phi(x) & x \in \mathbb{B}_1(Re_1), \\ \phi(x) = 0 & x \in \mathbb{B}_1^C(Re_1). \end{cases}$$

Let $\xi_0 = \min_{\mathbb{B}_1(0)} u$. Since u is positive, then $\xi_0 > 0$ and $m_0 = \frac{f(\xi_0)}{\sup_{R^n} u} > 0$. since u is monotone increasing in x_1 direction, it follows that

$$(-\triangle)^{\alpha/2} u(x) \geq a(R-1) m_0 u \quad \text{in } \mathbb{B}_1(Re_1)$$

It is assumed that $a(x_1) \to \infty$ as $x_1 \to \infty$, if one chooses $R > 0$ sufficiently large, we have

$$(-\triangle)^{\alpha/2} u(x) \geq \lambda_1 u(x), \quad \forall\, x \in \mathbb{B}_1(Re_1). \tag{28}$$

Let

$$m = \max_{\mathbb{B}_1(Re_1)} \frac{\phi}{u} \quad \text{and} \quad v(x) = m u(x).$$

Then obviously,

$$\begin{cases} (-\triangle)^{\alpha/2}(v(x) - \phi(x)) \geq 0 & \forall\, x \in \mathbb{B}_1(Re_1) \\ v(x) - \phi(x) > 0 & \forall\, x \in \mathbb{B}_1^C(Re_1). \end{cases}$$

By the *strong maximum principle*, we must have

$$v(x) > \phi(x), \quad \forall\, x \in \mathbb{B}_1(Re_1).$$

This contradicts the definition of v, because at a maximum point x^o, we have

$$v(x^o) = \frac{\phi(x^o)}{u(x^o)} u(x^o) = \phi(x^o).$$

Therefore, equation (27) does not possess any positive solution, and hence we complete the proof of Theorem 2.

4 Appendix

Here we prove Lemma 2.1. Without loss of generality, we may choose $a = 0$.

In the definition

$$(-\triangle)^{\alpha/2}(|x|^\gamma) = C_{n,\alpha} P.V. \int_{R^n} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+\alpha}} dy,$$

let $y = |x|z$, then it becomes

$$|x|^{\gamma-\alpha} C_{n,\alpha} P.V. \int_{R^n} \frac{1 - |z|^\gamma}{|\frac{x}{|x|} - z|^{n+\alpha}} dz = |x|^{\gamma-\alpha} C_{n,\alpha} P.V. \int_{R^n} \frac{1 - |z|^\gamma}{|e - z|^{n+\alpha}} dz = |x|^{\gamma-\alpha} C_\gamma.$$

Here e is any unit vector in R^n.

To see the sign of C_γ, we split the integral into two part, then make change of variable $z = \frac{y}{|y|^2}$ in the second part to arrive at

$$\begin{aligned} C_\gamma &= C_{n,\alpha} P.V. \left\{ \int_{\mathbb{B}_1(0)} \frac{1-|z|^\gamma}{|e-z|^{n+\alpha}} dz + \int_{\mathbb{B}_1(0)} \frac{1-|y|^{-\gamma}}{|e-y|^{n+\alpha}|y|^{n-\alpha}} dy \right\} \\ &= C_{n,\alpha} P.V. \int_{\mathbb{B}_1(0)} \frac{(1-|z|^\gamma)(1-|z|^{\alpha-n-\gamma})}{|e-z|^{n+\alpha}} dz. \end{aligned}$$

Now the conclusion of Lemma 2.1 follows immediately.

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