J. Differential Equations 252 (2012) 1589–1602

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Characterization of balls in terms of Bessel-potential integral equation ☆

Xiaolong Han [a], Guozhen Lu [b,a,*], Jiuyi Zhu [a]

[a] Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
[b] School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China

ARTICLE INFO

Article history:
Received 27 February 2011
Revised 7 July 2011
Available online 30 August 2011

MSC:
31B10
35N25

Keywords:
Overdetermined problem
Bessel potential
Moving plane method in integral form
Fractional differential equations
Characterizations of balls

ABSTRACT

For a bounded C^1 domain $\Omega \subset \mathbb{R}^N$, we consider the Bessel potential

$$u(x) = \int_{\Omega} g_\alpha(x-y)\,dy$$

for $2 \leqslant \alpha < N$. We show that $u =$ constant on $\partial\Omega$ if and only if Ω is a ball. More general Bessel-potential integral equation

$$u(x) = \int_{\Omega} g_\alpha(x-y)h\big(u(y)\big)\,dy$$

is also studied. Similar characterization of balls holds under certain assumptions on u and $h(u(y))$. We will use an integral form of the celebrated Alexandroff (1962) [2], Serrin (1971) [28], and Gidas, Ni and Nirenberg (1979) [16], (1981) [17] moving plane method developed by Chen, Li and Ou (2006) in [7] to establish our main results.

☆ Research is partly supported by a US NSF grant #DMS0901761.

* Corresponding author at: Department of Mathematics, Wayne State University, Detroit, MI 48202, USA.
E-mail addresses: xlhan@wayne.edu (X. Han), gzlu@math.wayne.edu (G. Lu), jiuyi.zhu@wayne.edu (J. Zhu).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.07.037

1. Introduction

A number of potential characterizations of balls are known in the literature. For instance, a single layer potential is given by

$$u(x) = \begin{cases} A \int_{\partial\Omega} \frac{-1}{2\pi} \log \frac{1}{|x-y|} \, d\sigma_y, & N = 2, \\ A \int_{\partial\Omega} \frac{1}{(N-2)\omega_N} \frac{1}{|x-y|^{N-2}} \, d\sigma_y, & N \geqslant 3, \end{cases} \tag{1.1}$$

where $A > 0$ is the constant source density on the boundary of the domain Ω. It is shown that if u is constant in $\bar{\Omega}$, then Ω can be proved to be a ball under different smoothness assumptions on the domain Ω. See [23] for the case of $N = 2$ and [26] for the case of $N \geqslant 3$. We also refer the reader to the book of C. Kenig [19] on this subject of layer potential.

It is also well known that the gravitational potential of a ball of constant mass density is constant on the surface of the ball. This property actually provides a characterization of balls as well. Indeed, Fraenkel [14] proves the following

Theorem A. *(See [14].) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and ω_N be the surface measure of the unit sphere in $\mathbb{R}^N$. Consider*

$$u(x) = \begin{cases} \frac{1}{2\pi} \int_{\Omega} \log \frac{1}{|x-y|} \, dy, & N = 2, \\ \frac{1}{(N-2)\omega_N} \int_{\Omega} \frac{1}{|x-y|^{N-2}} \, dy, & N \geqslant 3. \end{cases} \tag{1.2}$$

If $u(x)$ is constant on $\partial\Omega$, then Ω is a ball.

This result has been extended by Reichel [27] to more general Riesz potential, but under a more restrictive assumption on the domain Ω, i.e., Ω is assumed to be convex. In [27], Reichel considers the integral equation

$$u(x) = \begin{cases} \int_{\Omega} \log \frac{1}{|x-y|} \, dy, & \alpha = N, \\ \int_{\Omega} \frac{1}{|x-y|^{N-\alpha}} \, dy, & \alpha \neq N, \end{cases} \tag{1.3}$$

and proves the following theorem.

Theorem B. *(See [27].) Let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain and $\alpha > 2$, if $u(x)$ defined by (1.3) is constant on $\partial\Omega$, then Ω is a ball.*

Moreover, Reichel in [27] raised two open questions for (1.3):

Question 1. Is Theorem B true if we remove the convexity assumption of Ω?

Question 2. Is there an analogous result as Theorem B for the logarithmic Riesz potential

$$u(x) = \int_{\Omega} |x-y|^{\alpha-N} \log \frac{1}{|x-y|} \, dy? \tag{1.4}$$

Recently in [22], the authors answered the above two open questions to some extent. As for Question 1, instead of the convexity assumption on Ω, we only assume that Ω is a bounded C^1 domain for Theorem B to be true. As far as Question 2 is concerned, we proved in [22] that when Ω is a C^1 bounded domain with $\operatorname{diam}\Omega < e^{\frac{1}{N-\alpha}}$, then the analogous result holds.

It is known that u satisfying (1.3) is equivalent to saying that u satisfies the following fractional Laplacian equation

$$(-\triangle)^{\frac{\alpha}{2}} u(x) = \chi_\Omega(x) \tag{1.5}$$

in the sense of distribution. Therefore, our results in [22] also solve the overdetermined problem to the differential equation (1.5) when Ω is a bounded C^1 domain. Namely, if Ω is a bounded and open C^1 domain and u satisfying (1.5) is a constant on $\partial\Omega$, then Ω is a ball.

The overdetermined problems have attracted a lot of attention in the past decades. In his seminal paper [28], Serrin proved that overdetermined boundary problem characterizes the geometry of the underlying set. That is, if Ω is a bounded C^2 domain and $u \in C^2(\bar{\Omega})$ is a solution of

$$\begin{cases} \Delta u = -1 & \text{in } \bar{\Omega}, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = \text{const} & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

then Ω is a ball, moreover, u is radially symmetric with respect to its center of the ball.

Since the work of [28], there have been many generalizations to general equations. For instance, the overdetermined problem for the equation

$$\begin{cases} div\big(A\big(|\nabla u|\big)\nabla u\big) = -1 & \text{in } \bar{\Omega}, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = \text{constant} & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

was studied in [10–12,18] and references therein, under certain suitable assumptions. The interested reader may also refer to [3,9,4,25,31,13,32] and references therein, for other general elliptic equations. See also [1,26,29] and reference therein for overdetermined problems in an exterior domain or general domain.

In [4], more general elliptic equations

$$\begin{cases} S_k\big(D^2 u\big) = \dbinom{n}{k} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega,\ k \in \{1,\dots,n\}, \\ \dfrac{\partial u}{\partial n} = 1 & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

are considered, where $S_k(D^2u) = \binom{n}{k}$ is the k-elementary symmetry function of the eigenvalue of D^2u. Note that, when $k = 1$, it is the Poisson equation while $k = n$ it leads to the Monge–Ampère equation. The authors show that Ω is a ball and u is radially symmetric. The interested reader may refer to [12] and references therein, for quasilinear operator types of overdetermined problems.

There are many applications of overdetermined problems in mathematical physics. Many models in fluid mechanics, solid mechanics, thermodynamics, and electrostatics are relevant to the overdetermined Dirichlet or Neumann boundary problems of elliptic partial differential equations. Interested reader may refer to the article [10] for a nice introduction in that aspect.

In this paper, we consider the Bessel-potential type equation:

$$u(x) = \int_\Omega g_\alpha(x-y)\,dy, \tag{1.9}$$

where g_α is the Bessel kernel whose precise definition will be given in Section 2. Our main results are

Theorem 1. *Let Ω be a C^1 bounded domain. Assume that g_α is the Bessel kernel and u in* (1.9) *is constant on $\partial\Omega$, then Ω is a ball.*

Furthermore, we study more general Bessel-potential equation in bounded domains:

$$u(x) = \int_\Omega g_\alpha(x-y)h\big(u(y)\big)\,dy. \tag{1.10}$$

Then the following theorem is established.

Theorem 2. *Assume that the nonnegative solution $u(x) \in L^q(\Omega)$ for some $q > \frac{N}{N-\alpha}$, and $h(u)$ satisfies*:

(i) *$h(u)$ is continuous, increasing and $h(0) = 0$;*
(ii) *$h'(u)$ is non-decreasing and $h'(u) \in L^{r+1}$ for some $r > \frac{N}{\alpha}$.*

If $u(x)$ in (1.10) *is constant on the boundary of Ω, then Ω is a ball.*

Remark 1.1. Based on the assumption of (i) and (ii), we can infer that $\frac{h(u)}{u} \in L^{r+1}(\Omega)$ and $h'(u) \in L^{\frac{N}{\alpha}}(\Omega)$.

Remark 1.2. In the above two theorems, if the conclusion that Ω is a ball is verified, then we can easily deduce that $u(x)$ is radially symmetric with respect to the center of the ball.

Heuristically, (1.10) is closely related to the following fractional equation

$$(I-\Delta)^{\frac{\alpha}{2}} u(x) = h\big(u(x)\big)\chi_\Omega(x)$$

in the sense of distribution. In the case of $\alpha = 2$, it turns out to be the ground state of the Schrödinger equation. In [24], the symmetry property of the solutions of the Bessel-potential integral equation in $\mathbb{R}^N$ is shown.

Our approach is a new variant of moving plane method – moving plane in integral forms. The classical moving plane method based on maximum principle is developed in the pioneering works by Alexandroff [2], Serrin [28] and Gidas, Ni and Nirenberg [16,17]. See also Caffarelli, Gidas and Spruck [5], Chang and Yang [8], Wei and Xu [31], etc. Right after Serrin's paper, a short proof was presented by Weinberger [33] for the same result of [28].

The moving plane method in integral forms is much different from the traditional methods of moving planes used for partial differential equations. Instead of relying on the differentiability and maximum principles of the structure, a global integral norm is estimated. The method of moving planes in integral forms can be adapted to obtain symmetry and monotonicity for solutions. The method of moving planes on integral equations was developed in the work of [7], see also [20,8], the book [6] and an exhaustive list of references therein, where the symmetry of solutions in the entire space was proved. Moving plane method in integral form is also carried out in symmetry problems arising from the integral equations over bounded domains, see the work of [21].

We remark here that results for the Bessel potential proved in this paper and those for the Riesz potential or logarithmic Riesz potential in [22] can be extended to potentials corresponding to more general kernels as long as the kernels are monotone and satisfy some integrability conditions. More precisely, consider the integral equation for a bounded domain $\Omega \subset \mathbb{R}^N$, i.e.,

$$u(x) = \int_\Omega g\big(|x-y|\big)\,dy.$$

Assume that $g(r) \in C^1(\mathbb{R}_+)$ satisfies either

$$g'(r) < 0, \quad \forall 0 < r < \operatorname{diam}(\Omega),$$

or

$$g'(r) > 0, \quad \forall 0 < r < \operatorname{diam}(\Omega).$$

Moreover,

$$\epsilon^{-1} \int_0^{\epsilon} \left|g(r)\right| r^{N-1}\, dr \to 0$$

and

$$\int_0^{\epsilon} \left|g'(r)\right| r^{N-1}\, dr \to 0,$$

as $\epsilon \to 0$. Then we can show u is constant on $\partial\Omega$ if and only if Ω is a ball. We refer the reader to an updated version of [22] for more details.

The paper is organized as follows. In Section 2, we present some preliminaries on Bessel potential. In Section 3, we carry out the proof of Theorem 1. While in Section 4, we derive the proof of Theorem 2. Throughout this paper, the positive constant C is frequently used in the paper. It may differ from line to line, even within the same line. It also may depend on u in some cases.

2. Preliminaries on Bessel potential

In this section, we recall some basic properties of the Bessel potentials. The interested readers may refer to [15,30,34] for more details.

Definition 1. The Bessel kernel g_α with $\alpha \geqslant 0$ is defined by

$$g_\alpha(x) = \frac{1}{r(\alpha)} \int_0^{\infty} \exp\left(-\frac{\pi}{\delta}|x|^2\right) \exp\left(-\frac{\delta}{4\pi}\right) \delta^{\frac{\alpha-N-2}{2}}\, d\delta, \tag{2.1}$$

where $r(\alpha) = (4\pi)^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2})$.

For convenience, we set $G_\alpha(x,\delta) := \frac{1}{r(\alpha)} \exp(-\frac{\pi}{\delta}|x|^2) \exp(-\frac{\delta}{4\pi}) \delta^{\frac{\alpha-N-2}{2}}$.

Definition 2. The Bessel potentials $B_\alpha(f)$, $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, are given by

$$B_\alpha(f) = \begin{cases} g_\alpha * f, & \alpha > 0, \\ f, & \alpha = 0, \end{cases} \tag{2.2}$$

where $*$ denotes the convolution of functions.

Lemma 2.1. *For $0 < \alpha < N$, there exists C such that*

$$g_\alpha(x) \leqslant \frac{C}{|x|^{N-\alpha}} e^{-C|x|} \tag{2.3}$$

for all $x \in \mathbb{R}^N$. Moreover, it also follows that

$$\left|Dg_\alpha(x)\right| \leqslant \frac{C}{|x|^{N-\alpha+1}} e^{-C|x|}. \tag{2.4}$$

By using the L^p to L^q boundedness of the fractional integral (see [30]), the following Hardy–Littlewood–Sobolev type inequality can be easily derived:

Lemma 2.2. *Let $0 < \alpha < N$ and $1 < p < q < \infty$ satisfying $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{N}$. Then there exists constant C such that for all f in $L^p(\mathbb{R}^N)$, we have*

$$\left\|B_\alpha(f)\right\|_{L^q} \leqslant C\|f\|_{L^p}. \tag{2.5}$$

3. Proof of Theorem 1

In this section, we first introduce some notations. Choose any direction and, rotate coordinate system if it is necessary such that x_1-axis is parallel to it. For any $\lambda \in R$, define

$$T_\lambda = \left\{(x_1, \dots, x_N) \in \Omega \;\middle|\; x_1 = \lambda\right\}.$$

Since Ω is bounded, if λ is sufficiently negative, the intersection of T_λ and Ω is empty. Then, we move the plane T_λ all the way to the right until it intersects Ω. Let

$$\lambda_0 = \min\{\lambda\colon\ T_\lambda \cap \bar{\Omega} \neq \emptyset\}.$$

For $\lambda > \lambda_0$, T_λ cuts off Ω. We define

$$\Sigma_\lambda = \{x \in \Omega \mid x_1 < \lambda\}.$$

Set

$$x_\lambda = \{2\lambda - x_1, \dots, x_N\}$$

and

$$\Sigma'_\lambda = \{x_\lambda \in \Omega \mid x \in \Sigma_\lambda\}.$$

At the beginning of $\lambda > \lambda_0$, Σ'_λ remains within Ω. As the plane keeps moving to the right, Σ'_λ will still stay in Ω until at least one of the following events occurs:

(i) Σ'_λ is internally tangent the boundary of Ω at some point P_λ not on T_λ.
(ii) T_λ reaches a position where it is orthogonal to the boundary of Ω at some point Q.

Let $\bar{\lambda}$ be the first value such that at least one of above positions is reached.

We assert that Ω must be symmetric about $T_{\bar{\lambda}}$; i.e.,

$$\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega. \tag{3.1}$$

If this assertion is true, then for any given direction in $\mathbb{R}^N$, we can show that there also exists a plane $T_{\bar{\lambda}}$ such that Ω is symmetric about $T_{\bar{\lambda}}$. Since Ω is connected, then the only domain with those properties is a ball, see [2]. This thus proves our theorem.

We will first establish some lemmas. Throughout the paper we assume $2 \leqslant \alpha < N$.

Lemma 3.1. *u is in $C^1(\bar{\Omega})$ in* (1.9).

Remark. In fact, we can show that $u \in C^l(\mathbb{R}^N)$ for all $1 \leqslant l < \alpha$.

Proof of Lemma 3.1. Fix $\eta \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \leqslant \eta \leqslant 1$, and $\eta(t) = 0$ as $|t| \leqslant 1$, and $\eta(t) = 1$ as $|t| \geqslant 2$. Define for any ϵ,

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(y) g_\alpha(x-y)\,dy,$$

where $\eta_\epsilon = \eta(\frac{|x-y|}{\epsilon})$. Thanks to (2.3) and (2.4), we can easily deduce that

$$u_\epsilon \to u;$$
$$D_{x_i} u_\epsilon \to \int_\Omega D_{x_i} g_\alpha(x-y)\,dy$$

uniformly in Ω as $\epsilon \to 0$. Therefore, the lemma follows. □

In order to assert (3.1), we introduce

$$u_\lambda(x) = u(x_\lambda),$$
$$\Omega_\lambda = \Omega \setminus \left(\overline{\Sigma_\lambda \cup \Sigma'_\lambda}\right).$$

Lemma 3.2. *For $\lambda_0 < \lambda < \bar{\lambda}$ and $u(x)$ in* (1.9)*, $u(x) < u_\lambda(x)$ for any $x \in \Sigma_\lambda$.*

Proof. For any $x \in \Sigma_\lambda$, we rewrite $u(x)$ and $u(x_\lambda)$ as

$$u(x) = \int_{\Sigma_\lambda} g_\alpha(x-y)\,dy + \int_{\Sigma_\lambda} g_\alpha(x_\lambda - y)\,dy + \int_{\Omega_\lambda} g_\alpha(x-y)\,dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} g_\alpha(x_\lambda - y)\,dy + \int_{\Sigma_\lambda} g_\alpha(x-y)\,dy + \int_{\Omega_\lambda} g_\alpha(x_\lambda - y)\,dy.$$

Then

$$u(x) - u_\lambda(x) = \int_{\Omega_\lambda} \left[g_\alpha(x-y) - g_\alpha(x_\lambda - y)\right] dy. \tag{3.2}$$

Since $|x-y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$, due to the explicit formula of g_α, $g_\alpha(x-y) < g_\alpha(x_\lambda - y)$, which implies

$$u_\lambda(x) > u(x)$$

and then the lemma holds. □

Lemma 3.3. *$u(x)$ satisfies (1.9) and suppose $\lambda = \bar\lambda$ in the first case; i.e., Σ'_λ is internally tangent to the boundary of Ω at some point $P_{\bar\lambda}$ not on $T_{\bar\lambda}$, then $\Sigma_{\bar\lambda} \cup T_{\bar\lambda} \cup \Sigma'_{\bar\lambda} = \Omega$.*

Proof. Thanks to Lemma 3.2, $u_{\bar\lambda}(x) \geqslant u(x)$ for $x \in \Sigma_{\bar\lambda}$. We argue by contradiction. Suppose $\Sigma_{\bar\lambda} \cup T_{\bar\lambda} \cup \Sigma'_{\bar\lambda} \subsetneqq \Omega$; that is, $\Omega_{\bar\lambda} \neq \emptyset$. At P, from (3.2), $u(P_{\bar\lambda}) > u(P)$. It is a contradiction since $P_{\bar\lambda}, P \in \partial\Omega$ and $u(P_{\bar\lambda}) = u(P) =$ constant. Therefore, the lemma is completed. □

Lemma 3.4. *$u(x)$ satisfies (1.9) and suppose that the second case occurs: i.e., $T_{\bar\lambda}$ reaches a position where is orthogonal to the boundary of Ω at some point Q, then $\Sigma_{\bar\lambda} \cup T_{\bar\lambda} \cup \Sigma'_{\bar\lambda} = \Omega$.*

Proof. Since $u(x)$ is constant on the boundary and $\Omega \in C^1$, ∇u is parallel to the normal at Q. As implied in the second case, $\frac{\partial u}{\partial x_1}|_Q = 0$. We denote the coordinate of Q by z. Suppose $\Omega_{\bar\lambda} \neq \emptyset$, there exists a ball $B \Subset \Omega_{\bar\lambda}$. Choosing a sequence $\{x^i\}_1^\infty \in \Sigma_{\bar\lambda} \setminus T_{\bar\lambda}$ such that $x^i \to z$ as $i \to \infty$. It is easy to see that $x^i_{\bar\lambda} \to z$ as $i \to \infty$. Since $B \Subset \Omega_{\bar\lambda}$, we can also find a $\tau > 0$ such that $\operatorname{diam}\Omega > |x^i_{\bar\lambda} - y| > \tau$ for any $y \in B$ and any $x^i_{\bar\lambda}$.

From (3.2) and (2.1), choosing a unit vector $e_1 = (1, 0, \dots, 0)$, by Mean Value Theorem,

$$\begin{aligned}
\frac{u(x^i) - u(x^i_{\bar\lambda})}{(x^i - x^i_{\bar\lambda}) \cdot e_1} &= \int_{\Omega_{\bar\lambda}} \int_0^\infty \frac{G_\alpha(x^i - y, \delta) - G_\alpha(x^i_{\bar\lambda} - y, \delta)}{(x^i - x^i_{\bar\lambda}) \cdot e_1}\, d\delta\, dy \\
&= \int_{\Omega_{\bar\lambda}} \int_0^\infty G_\alpha(\xi - y, \delta)\left(-\frac{2\pi}{\delta}(\xi - y) \cdot e_1\right) d\delta\, dy \\
&> \int_B \int_0^\infty G_\alpha(\xi - y, \delta)\left(-\frac{2\pi}{\delta}(\xi - y) \cdot e_1\right) d\delta\, dy \\
&> C,
\end{aligned} \tag{3.3}$$

where ξ is some point between $x^i_{\bar\lambda}$ and x^i. Nevertheless,

$$\lim_{i\to\infty} \frac{u(x^i_{\bar\lambda}) - u(x^i)}{(x^i_{\bar\lambda} - x^i) \cdot e_1} = \frac{\partial u}{\partial x_1}\bigg|_Q = 0,$$

which contradicts (3.3). Therefore, $\Omega_{\bar\lambda} = \emptyset$. □

Combining Lemmas 3.3 and 3.4, Theorem 1 is proved.

4. Proof of Theorem 2

We show that the assumptions in Theorem 2 imply $u \in C^1(\bar\Omega)$. First we introduce a regularity lifting lemma in [6].

Lemma 4.1 *(Regularity lifting). Let V be a Hausdorff topological vector space. Suppose there are two extended norms (i.e. the norm of an element in V might be infinity) defined on V,*

$$\|\cdot\|_X, \|\cdot\|_Y : V \to [0, \infty].$$

Assume that the spaces

$$X := \{ v \in V\colon \|v\|_X < \infty \} \quad \text{and} \quad Y := \{ v \in V\colon \|v\|_Y < \infty \}$$

are complete under the corresponding norms, and the convergence in X or in Y implies the convergence in V.

Let T be a contracting map from X into itself and from Y into itself. Assume that $f \in X$, and that there exists a function $g \in Z := X \cap Y$ such that $f = Tf + g$ in X. Then f also belongs to Z.

Lemma 4.2. *If u, $h(u)$ satisfy the assumptions in Theorem 2, then $u \in C^1(\bar{\Omega})$.*

Proof. Define the linear operator

$$T_u v = \int_{\Omega} g_\alpha(x-y) \frac{h(u)}{u} v \, dy.$$

For any real number $a > 0$, set

$$\begin{cases} u_a(x) = u(x), & |u(x)| > a, \\ u_a(x) = 0, & \text{if otherwise}. \end{cases}$$

Let $u_b(x) = u(x) - u_a(x)$.

Since $u(x)$ satisfies (1.10), we can write it as

$$u_a(x) = T_{u_a} u_a + g(x) - u_b(x) \tag{4.1}$$

with $g(x) = \int_\Omega g_\alpha(x-y) h(u_b)\, dy$.

Due to the continuity of $h(u)$ and Lemma 2.1, $g(x) \in L^\infty(\Omega)$.

As for $T_{u_a} v$, apply Lemma 2.2, then *Hölder*'s inequality again, for any $t > \frac{N}{N-\alpha}$,

$$\begin{aligned} \|T_{u_a} v\|_{L^t(\Omega)} &\leqslant C \left\| \frac{h(u_a)}{u_a} v \right\|_{L^{\frac{Nt}{N+\alpha t}}(\Omega)} \\ &\leqslant C \left\| \frac{h(u_a)}{u_a} \right\|_{L^{\frac{N}{\alpha}}(\Omega)} \|v\|_{L^t(\Omega)}. \end{aligned}$$

Choose $a > 0$ sufficiently large, then

$$\|T_{u_a} v\|_{L^t(\Omega)} \leqslant \frac{1}{2} \|v\|_{L^t(\Omega)}.$$

Therefore, T_{u_a} is a contracting map. By the regularity lifting lemma above, $u_a \in L^t \cap L^q$ for any $t > \frac{N}{N-\alpha}$. This also implies that $u \in L^m(\Omega)$ for any $m \geqslant 1$.

Next we show that $u \in L^\infty(\Omega)$. For any $x \in \Omega$, choose a ball $\mathbb{B}_R(x)$ with fixed radius R such that $\Omega \Subset \mathbb{B}_R(x)$, then by *Hölder*'s inequality,

$$\begin{aligned}
\left|u(x)\right| &\leqslant \left|\int_\Omega \frac{1}{|x-y|^{N-\alpha}} h(u)\,dy\right| \\
&\leqslant \left\| |x-y|^{\alpha-N} \right\|_{L^{\frac{r}{r-1}}(\mathbb{B}_R(x))} \left\| \frac{h(u)}{u} \right\|_{L^{r+1}(\Omega)} \|u\|_{L^{r(r+1)}(\Omega)} \\
&\leqslant C,
\end{aligned}$$

due to the fact that $r > \frac{N}{\alpha}$ implies that $-(N-\alpha)\frac{r}{r-1} + N > 0$ and $r(r+1) > \frac{N}{N-\alpha}$, the assumption of $h(u)$, and the fact that $u \in L^m$ for any $m > 0$. Thanks to the continuity of $h(u)$, furthermore, we can infer that $h(u) < C$. We claim that $u \in C^1(\bar{G})$. Fix $\eta \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \leqslant \eta \leqslant 1$, and $\eta(t) = 0$ as $|t| \leqslant 1$, and $\eta(t) = 1$ as $|t| \geqslant 2$. Define for any ϵ,

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(y) g_\alpha(x-y) h\big(u(y)\big)\,dy,$$

where $\eta_\epsilon(y) = \eta(\frac{|x-y|}{\epsilon})$. As in Lemma 3.1, we can deduce that

$$\begin{gathered}
u_\epsilon \to u; \\
D_{x_i} u_\epsilon \to \int_\Omega D_{x_i} g_\alpha(x-y) h(u)\,dy
\end{gathered}$$

uniformly in Ω as $\epsilon \to 0$. Therefore, we have verified the claim. The lemma follows. □

Since $|x_\lambda - y_\lambda| = |x-y|$ and $|x_\lambda - y| = |x - y_\lambda|$, for any solution in (1.10), we rewrite $u(x)$ and $u_\lambda(x)$ in the following forms:

$$u(x) = \int_{\Sigma_\lambda} g_\alpha(x-y) h\big(u(y)\big)\,dy + \int_{\Sigma_\lambda} g_\alpha(x_\lambda - y) h\big(u_\lambda(y)\big)\,dy + \int_{\Omega_\lambda} g_\alpha(x-y) h\big(u(y)\big)\,dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} g_\alpha(x_\lambda - y) h\big(u(y)\big)\,dy + \int_{\Sigma_\lambda} g_\alpha(x-y) h\big(u_\lambda(y)\big)\,dy + \int_{\Omega_\lambda} g_\alpha(x_\lambda - y) h\big(u(y)\big)\,dy.$$

Then,

$$\begin{aligned}
u(x) - u_\lambda(x) = &\int_{\Sigma_\lambda} \big(g_\alpha(x-y) - g_\alpha(x_\lambda - y)\big)\big(h\big(u(y)\big) - h\big(u_\lambda(y)\big)\big)\,dy \\
&+ \int_{\Omega_\lambda} \big(g_\alpha(x-y) - g_\alpha(x_\lambda - y)\big) h\big(u(y)\big)\,dy. \qquad (4.2)
\end{aligned}$$

Since $g_\alpha(x-y) < g_\alpha(x_\lambda - y)$ for $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$, then

$$u(x) - u_\lambda(x) \leqslant \int_{\Sigma_\lambda} \big(g_\alpha(x-y) - g_\alpha(x_\lambda - y)\big)\big(h\big(u(y)\big) - h\big(u_\lambda(y)\big)\big)\,dy. \qquad (4.3)$$

In order to start to move the plane, we define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid u(x) > u_\lambda(x)\},$$

and

$$w_\lambda(x) = u(x) - u_\lambda(x).$$

Lemma 4.3. *For λ sufficiently close to λ_0, assume $u(x)$ satisfies* (1.10)*, then $u(x) \leqslant u_\lambda(x)$ for any $x \in \Sigma_\lambda$.*

Proof. Due to the fact that $g_\alpha(x-y) > g_\alpha(x_\lambda - y)$ for $x \in \Sigma_\lambda$ and $y \in \Sigma_\lambda$, from (4.3)

$$\begin{aligned} u(x) - u_\lambda(x) &\leqslant \int_{\Sigma_\lambda^-} \big(g_\alpha(x-y) - g_\alpha(x_\lambda - y)\big)\big(h\big(u(y)\big) - h\big(u_\lambda(y)\big)\big)\,dy \\ &\leqslant \int_{\Sigma_\lambda^-} g_\alpha(x-y)\big(h\big(u(y)\big) - h\big(u_\lambda(y)\big)\big)\,dy \\ &= \int_{\Sigma_\lambda^-} g_\alpha(x-y) h'\big(\theta u + (1-\theta)u_\lambda\big)(u - u_\lambda)\,dy, \end{aligned} \tag{4.4}$$

where $h'(\theta u + (1-\theta)u_\lambda)$ is deduced by Mean Value Theorem and $0 < \theta < 1$.

Applying the estimate in (2.5), then *Hölder*'s inequality above, we get

$$\begin{aligned} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)} &\leqslant C\big\|h'\big(\theta u + (1-\theta)u_\lambda\big)w_\lambda\big\|_{L^{\frac{nq}{N+\alpha q}}(\Sigma_\lambda^-)} \\ &\leqslant C\big\|h'\big(\theta u + (1-\theta)u_\lambda\big)\big\|_{L^{\frac{N}{\alpha}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \end{aligned}$$

By the assumption (ii) of h, if λ is close enough to λ_0, then,

$$C\big\|h'\big(\theta u + (1-\theta)u_\lambda\big)\big\|_{L^{\frac{N}{\alpha}}(\Sigma_\lambda^-)} \leqslant \frac{1}{2},$$

which implies that

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0.$$

Hence Σ_λ^- measures 0, furthermore, $w_\lambda(x) \leqslant 0$ for any $x \in \Sigma_\lambda$. The lemma is completed. □

Lemma 4.4. *Suppose $\lambda < \bar\lambda$ and $u(x) \leqslant u_\lambda(x)$ in Σ_λ, then there exists $\epsilon > 0$ such that $u(x) < u_{\hat\lambda}(x)$ for any $x \in \Sigma_{\hat\lambda}$, where $\bar\lambda > \hat\lambda := \lambda + \epsilon$.*

Proof. Since $u(x) \leqslant u_\lambda(x)$, then $h(u) \leqslant h(u_\lambda)$ by the assumption of h. Suppose there exists some point x^0 in Σ_λ such that $u(x_\lambda^0) - u(x^0) = 0$; that is, from (4.2),

$$0 = \int_{\Sigma_\lambda} \big[g_\alpha\big(x^0 - y\big) - g_\alpha\big(x_\lambda^0 - y\big)\big]\big[h(u) - h(u_\lambda)\big]\,dy + \int_{\Omega_\lambda} \big[g_\alpha\big(x^0 - y\big) - g_\alpha\big(x_\lambda^0 - y\big)\big]h(u)\,dy.$$

Thus, $h(u) \equiv 0$ in Ω_λ, which is impossible since $h(u) > 0$. Therefore, $u(x) < u_\lambda(x)$ in Σ_λ.

We next show that the plane T_λ can be moved a little further. Since $h'(u) \in L^{\frac{N}{\alpha}}(\Omega)$, by the continuity of integration, for any μ, there exists δ such that, for any measurable set E with $|E| < \delta$, then

$$\left\|h'(u)\right\|_{L^{\frac{N}{\alpha}}(E)} < \mu.$$

As shown in proving that $w_\lambda(x) < 0$ in Σ_λ, we choose a compact set $D \Subset \Sigma_\lambda$ such that $w_{\hat{\lambda}}(x) < 0$ in D. Thus the set $\Sigma_{\hat{\lambda}}^-$ can only lie in $F := \{\Sigma_\lambda \setminus D\} \cup \{\Sigma_{\hat{\lambda}} \setminus \Sigma_\lambda\}$. From (4.4),

$$w_{\hat{\lambda}} \leqslant \int_F g_\alpha(x-y)h'\big(\theta u + (1-\theta)u_{\hat{\lambda}}\big)w_{\hat{\lambda}}\,dy. \tag{4.5}$$

Choosing ϵ small enough and D appropriately large so that $|F| < \delta$, we have

$$\left\|h'\big(\theta u + (1-\theta)u_{\hat{\lambda}}\big)\right\|_{L^{\frac{n}{\alpha}}(F)} < \mu. \tag{4.6}$$

Employing Lemma 2.2 to (4.5), we get

$$\|w_{\hat{\lambda}}\|_{L^q(F)} \leqslant C\left\|h'\big(\theta u + (1-\theta)u_\lambda\big)w_{\hat{\lambda}}\right\|_{L^{\frac{Nq}{N+\alpha q}}(F)}.$$

Then, by *Hölder*'s inequality,

$$\|w_{\hat{\lambda}}\|_{L^q(F)} \leqslant C\left\|h'\big(\theta u + (1-\theta)u_{\hat{\lambda}}\big)\right\|_{L^{\frac{N}{\alpha}}(F)}\|w_{\hat{\lambda}}\|_{L^q(F)}.$$

Thanks to (4.6), if μ is small enough, $\|w_{\hat{\lambda}}\|_{L^q(F)} = 0$. Then, $\Sigma_{\hat{\lambda}}^-$ is empty; that is, $u(x) \leqslant u_{\hat{\lambda}}(x)$. Using the same argument at the beginning of the lemma, we shall show that $u(x) < u_{\hat{\lambda}}(x)$ for any $x \in \Sigma_{\hat{\lambda}}$. Therefore, the lemma holds. □

Lemma 4.5. *Suppose $u(x)$ satisfies* (1.10) *and $\lambda = \bar{\lambda}$ in the first case; i.e., $\Sigma'_{\bar{\lambda}}$ is internally tangent to the boundary of Ω at some point $P_{\bar{\lambda}}$ not on $T_{\bar{\lambda}}$, then $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega$.*

Proof. If not, then $\Omega_\lambda \neq \emptyset$. From (4.2), at P, $u(P) < u(P_{\bar{\lambda}})$ since $h(u_{\bar{\lambda}}) > h(u)$ in $\Sigma_{\bar{\lambda}}$ and $h(u) > 0$ in Ω_λ. However, $u(P) = u(P_{\bar{\lambda}})$ by our assumption that u is constant on $\partial\Omega$. Therefore, a contradiction is derived. Hence $\Omega_\lambda = \emptyset$, which implies that $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega$. □

Lemma 4.6. *$u(x)$ satisfies* (1.10) *and suppose that the second case occurs; i.e., $T_{\bar{\lambda}}$ reaches a position where is orthogonal to the boundary of Ω at some point Q, then, $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega$.*

Proof. As deduced before, $\frac{\partial u}{\partial x_1}|_Q = 0$. Denote the coordinate Q by z. Suppose $\Omega_{\bar{\lambda}} \neq \emptyset$, then there exists a ball $B \Subset \Omega_{\bar{\lambda}}$. Choose a sequence $\{x^i\}_1^\infty \in \Sigma_{\bar{\lambda}} \setminus T_{\bar{\lambda}}$ such that $x^i \to z$ as $i \to \infty$. Correspondingly $x^i_{\bar{\lambda}} \to z$ as $i \to \infty$. Since $B \Subset \Omega_{\bar{\lambda}}$, we can find a τ such that $|x^i_{\bar{\lambda}} - y| > \tau$ for any $y \in B$ and any $x^i_{\bar{\lambda}}$. By (4.2),

$$\begin{aligned}
\frac{u(x^i) - u(x^i_{\bar{\lambda}})}{(x^i - x^i_{\bar{\lambda}}) \cdot e_1} &\geqslant \int_{\Omega_{\bar{\lambda}}} \frac{g_\alpha(x^i - y) - g_\alpha(x^i_{\bar{\lambda}} - y)}{(x^i - x^i_{\bar{\lambda}}) \cdot e_1} h(u)\,dy \\
&= \int_{\Omega_{\bar{\lambda}}} \int_0^\infty G_\alpha(\xi - y, \delta)\left(-\frac{2\pi}{\delta}(\xi - y) \cdot e_1\right)h(u)\,d\delta\,dy
\end{aligned}$$

$$
\begin{aligned}
&> \int_B \int_0^\infty G_\alpha(\xi - y, \delta)\left(-\frac{2\pi}{\delta}\big((\xi - y)\cdot e_1\big)\right) h(u)\, d\delta\, dy \\
&> C. \qquad (4.7)
\end{aligned}
$$

As before, ξ is some point between $x^i_{\bar\lambda}$ and x^i and Mean Value Theorem is used above. However,

$$
\lim_{i\to\infty} \frac{u(x^i_{\bar\lambda}) - u(x^i)}{(x^i_{\bar\lambda} - x^i)\cdot e_1} = \frac{\partial u}{\partial x_1}\bigg|_Q = 0.
$$

It apparently contradicts (4.7). In the end, the lemma holds. $\square$

With the help of Lemmas 4.5 and 4.6, Theorem 2 is confirmed.

Acknowledgments

The authors wish to thank the referee for his very careful reading of the paper and for his kind comments on our work and constructive suggestions to improve the exposition of the paper. Part of the work was done while the second author was visiting Beijing Normal University whose hospitality is acknowledged.

References

[1] A. Aftalion, J. Busca, Radial symmetry of overdetermined boundary-value problems in exterior domains, Arch. Ration. Mech. Anal. 143 (2) (1998) 195–206.
[2] A.D. Alexandroff, A characteristic property of the sphere, Ann. Mat. Pura Appl. 58 (1962) 303–354.
[3] A. Bennett, Symmetry in an overdetermined four order elliptic boundary value problem, SIAM J. Math. Anal. 17 (1986) 1354–1358.
[4] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Serrin-type overdetermined problems: an alternative proof, Arch. Ration. Mech. Anal. 190 (2008) 267–280.
[5] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. XLII (1989) 271–297.
[6] W. Chen, C. Li, Methods on Nonlinear Elliptic Equations, AIMS Ser. Differ. Equ. Dyn. Syst., vol. 4, AIMS, 2010.
[7] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59 (2006) 330–343.
[8] S.Y.A. Chang, P.C. Yang, On uniqueness of solutions of nth order differential equations in conformal geometry, Math. Res. Lett. 4 (1997) 91–102.
[9] A. Cianchi, P. Salani, Overdetermined anisotropic elliptic problems, Math. Ann. 345 (4) (2009) 859–881.
[10] I. Fragala, F. Gazzola, Partially overdetermined elliptic boundary value problems, J. Differential Equations 245 (2008) 1299–1322.
[11] I. Fragala, F. Gazzola, B. Kawohl, Overdetermined problems with possibly degenerate ellipticity, a geometric approach, Math. Z. 254 (2006) 117–132.
[12] A. Farina, B. Kawohl, Remarks on an overdetermined boundary value problem, Calc. Var. Partial Differential Equations 31 (3) (2008) 351–357.
[13] A. Farina, E. Valdinoci, Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems, Arch. Ration. Mech. Anal. 195 (2010) 1025–1058.
[14] L.E. Fraenkel, Introduction to Maximum Principles and Symmetry in Elliptic Problems, Cambridge Tracts in Math., vol. 128, Cambridge University Press, London, 2000.
[15] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education, Inc., Upper Saddle River, NJ, 2004, xii+931 pp.
[16] B. Gidas, W. Ni, L. Nirenberg, Symmetry of related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.
[17] B. Gidas, W. Ni, L. Nireberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, in: Mathematical Analysis and Applications, Part A, in: Adv. Math. Suppl. Stud., vol. 7A, Academic Press, New York, 1981, pp. 369–402.
[18] N. Garofalo, J.L. Lewis, A symmetry result related to some overdetermined boundary value problems, Amer. J. Math. 111 (1989) 9–33.
[19] C. Kenig, Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Reg. Conf. Ser. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1994.
[20] Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. (JEMS) 6 (2004) 153–180.

[21] D. Li, G. Strohmer, L. Wang, Symmetry of integral equations on bounded domain, Proc. Amer. Math. Soc. 137 (2009) 3695–3702.
[22] G. Lu, J. Zhu, An overdetermined problem in Riesz-potential and fractional Laplacian, arXiv:1101.1649.
[23] E. Martensen, Eine Integralgleichung für die log. Gleichgewichtwverteilung und die Krümmung der Randkurve eines Gebies, ZAMM Z. Angew. Math. Mech. 72 (1992) 596–599.
[24] L. Ma, D. Chen, Radial symmetry and monotonicity for an integral equation, J. Math. Anal. Appl. 342 (2009) 943–949.
[25] L. Payne, G. Philippin, Some overdetermined boundary value problems for harmonic functions, Z. Angew. Math. Phys. 42 (6) (1991) 864–873.
[26] W. Reichel, Radial symmetry for elliptic boundary value problems on exterior domain, Arch. Ration. Mech. Anal. 137 (1997) 381–394.
[27] W. Reichel, Characterization of balls by Riesz-potentials, Ann. Mat. 188 (2009) 235–245.
[28] J. Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. Anal. 43 (1971) 304–318.
[29] B. Sirakov, Symmetry for exterior elliptic problems and two conjectures in potential theory, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001) 135–156.
[30] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Ser. Appl. Math., vol. 32, Princeton Univ. Press, Princeton, NJ, 1970.
[31] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 313 (2) (1999) 207–228.
[32] G. Wang, C. Xia, A characterization of the Wulff shape by an overdetermined anisotropic PDE, Arch. Ration. Mech. Anal. (2010).
[33] H. Weinberger, Remark on the preceding paper of Serrin, Arch. Ration. Mech. Anal. 43 (1971) 319–320.
[34] W. Ziemer, Weakly Differentiable Function: Sobolev Spaces and Function of Bounded Variation, Geom. Topol. Monogr., vol. 120, Springer-Verlag, New York, 1989.