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AXIAL SYMMETRY AND REGULARITY OF SOLUTIONS TO AN INTEGRAL EQUATION IN A HALF-SPACE

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We consider the integral equation

$$u(x) = \int_{\mathbb{R}^n_+} G(x, y) f(u(y))\, dy,$$

where $G(x, y)$ is the Green's function of the corresponding polyharmonic Dirichlet problem in a half-space. We prove by the method of moving planes in integral form that, under some integrability conditions, the solutions are axially symmetric with respect to some line parallel to the x_n-axis and non-decreasing in the x_n direction, which further implies the nonexistence of solutions. We also show similar results for a class of systems of integral equations. This appears to be the first paper in which the moving plane method in integral form is employed in a half-space to derive axial symmetry.

We also obtain the regularity of the integral equation in a half-space

$$u(x) = \int_{\mathbb{R}^n_+} G(x, y) |u(y)|^{p-1} u(y)\, dy$$

by the regularity lifting method. As a corollary, we prove the nonexistence of nonnegative solutions to this equation. Moreover, we show that the nonnegative solutions in this equation only depend on x_n if $u \in L^{2n/(n-2m)}_{\mathrm{loc}}(\mathbb{R}^n_+)$ and $1 < p < (n + 2m)/(n - 2m)$.

1. Introduction

We study the properties of nonnegative solutions of the following integral equations in a half-space:

$$u(x) = \int_{\mathbb{R}^n_+} G(x, y) f(u)\, dy, \tag{1}$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$, $n > 2m$ is the dimension of the half-space, and $G(x, y)$ is the Green's functions of $(-\Delta)^m$ related to a Dirichlet boundary condition in $\mathbb{R}^n_+$

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that will be presented in Section 2. We also assume that $f(u(x))$ satisfies

(f_1) $f : [0, \infty) \to \mathbb{R}$ is increasing, $f(0) \geq 0$,

and one of the following:

(f_2) $|\partial f(u)/\partial u| \leq C_1|u|^{\beta_1} + C_2|u|^{\beta_2}$, where $u^{\beta_1}, u^{\beta_2} \in L^{n/(2m)}(\mathbb{R}^n_+)$, β_1 is some nonnegative constant, β_2 is some nonpositive constant and C_1, C_2 are nonnegative constants, or

$(\tilde{f_2})$ $f'(\,\cdot\,)$ is nondecreasing and $f'(u) \in L^{n/(2m)}(\mathbb{R}^n_+)$.

Obviously, we should assume that $f \not\equiv 0$.

We prove the following theorem by the method of moving planes in integral form.

Theorem 1. *If* (f_1) *and either* (f_2) *or* $(\tilde{f_2})$ *hold, and the function* $u \in L^r(\mathbb{R}^n_+)$ *for some* $r > n/(n-2m)$ *is a nonnegative solution of* (1), *then* u *is trivial.*

The integral equation (1) is closely related to the higher order elliptic equations

$$\begin{cases} (-\Delta)^m u = f(u) & \text{in } \mathbb{R}^n_+, \\ u = \dfrac{\partial u}{\partial x_n} = \cdots = \dfrac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0 & \text{on } x_n = 0. \end{cases} \tag{2}$$

The weak solutions of (2) in the Sobolev space $H_0^m(\mathbb{R}^n_+)$ satisfy

$$\langle u, v\rangle_m = \int_{\mathbb{R}^n_+} f(u(x))v(x)\,dx \quad \text{for all } v \in H_0^m(\mathbb{R}^n_+), \tag{3}$$

where

$$\langle u, v\rangle_m = \begin{cases} \displaystyle\int_{\mathbb{R}^n_+} \Delta^{m/2}u\Delta^{m/2}v\,dx & m \text{ even}, \\ \displaystyle\int_{\mathbb{R}^n_+} \nabla\Delta^{(m-1)/2}u\nabla\Delta^{(m-1)/2}v\,dx & m \text{ odd}. \end{cases}$$

Under the conditions that $u \in C^{2m-1}(\bar{\mathbb{R}}^n_+) \cap W^{2m,p}_{\text{loc}}(\mathbb{R}^n_+)$ for some $p > (n)/2m$ and u and all partial derivatives of u of order less than or equal to $2m-1$ are bounded, Reichel and Weth [2009] prove that (2) implies (1). Therefore, Theorem 1 implies the following:

Corollary 1. *Suppose* (f_1) *and either* (f_2) *or* $(\tilde{f_2})$ *hold,* $p > n/(2m)$, $u \in L^r(\mathbb{R}^n_+)$ *for some* $r > n/(n-2m)$ *and* $u \in C^{2m-1}(\bar{\mathbb{R}}^n_+) \cap W^{2m,p}_{\text{loc}}(\mathbb{R}^n_+)$ *is a nonnegative solution of* (2). *Moreover, suppose that* u *and all the partial derivatives of* u *of order less than or equal to* $2m-1$ *are bounded. Then* u *is trivial.*

We also consider the properties of nonnegative solutions of the following systems of integral equations:

$$u_i(x) = \int_{\mathbb{R}^n_+} G(x, y) f_i(y', y_n, u_1, \ldots, u_N))\,dy, \quad i = 1, \ldots, N, \tag{4}$$

where $y' = (y_1, \dots, y_{n-1})$. Assume $f_i(x', x_n, u_1, \dots, u_N)$ $(i = 1, \dots, N)$ is nondecreasing with respect to every u_j $(j = 1, \dots, N)$, nonincreasing with respect to $|x'|$ and nondecreasing with respect to x_n. Moreover, assume that f_i satisfies

(f_i^1) $f_i : [0, \infty) \to \mathbb{R}$, $f_i(x', x_n, 0) \geq 0$,

and either of the following:

(f_i^2) $|\partial f_i / \partial u_j| \leq C_{ij}^0 \|u\|^{\beta_{ij}^0} + C_{ij}^1 \|u\|^{\beta_{ij}^1} + g_{ij}(x)$, or

$(\tilde{f}_i^2)$ $\partial f_i / \partial u_j$ is nondecreasing with respect to u_j and $\partial f_i / \partial u_j \in L^{n/(2m)}(\mathbb{R}_+^n)$.

In these conditions, $\|u\| := \sqrt{u_1^2 + \cdots + u_N^2}$ and we have assumed that

$$\|u\|^{\beta_{ij}^0}, \ \|u\|^{\beta_{ij}^1}, \ g_{ij} \in L^{n/(2m)}(\mathbb{R}_+^n),$$

C_{ij}^0 and C_{ij}^1 are nonnegative constants, β_{ij}^0 is a nonnegative constant and β_{ij}^1 is a nonpositive constant. Obviously, we also assume that $f_i \not\equiv 0$ for $i = 1, \dots, N$.

Definition (see [Jin and Li 2006]). Functions $f_1, \dots, f_N$ are *essentially related* if

$$\sum_{l=1}^{l_0} f_{i_l}(x', x_n, u_1, u_2, \dots, u_N) \neq \sum_{l=1}^{l_0} f_{i_l}(x', x_n, v_1, v_2, \dots, v_N),$$

provided that $u_i \leq v_i$ for $i = 1, \dots, N$ and $u_j < v_j$ for $j \in S$, where $S = \{1, \dots, N\} \setminus \{i_1, \dots, i_{l_0}\}$.

With this definition, the systems cannot be divided into independent subsystems.

We prove the following theorem with the moving plane method in integral form.

Theorem 2. *If* (f_i^1) *and either* (f_i^2) *or* $(\tilde{f}_i^2)$ *hold and the function* $u_i \in L^r(\mathbb{R}_+^n)$ *for some* $r > n/(n-2m)$ *is a nonnegative solution of* (4), *then* u_i *is trivial for* $i = 1, \dots, N$.

Similarly, the system (4) of integral equations is closely related to following system of higher order elliptic equations:

$$\begin{cases} (-\Delta)^m u_i = f_i(x', x_n, u_1, \dots, u_N) & \text{in } \mathbb{R}_+^n, \ i = 1, \cdots, N, \\ u_i = \dfrac{\partial u_i}{\partial x_n} = \cdots = \dfrac{\partial^{m-1} u_i}{\partial x_n^{m-1}} = 0 & \text{on } x_n = 0. \end{cases} \tag{5}$$

It is well known that the moving plane method was first developed by the Soviet mathematician Alexandrov in the 1950s. It was further developed by Serrin [1971], Gidas, Ni and Nirenberg [1979], Caffarelli, Gidas and Spruck [1989], Chen and Li [1991], Chang and Yang [1997], Wei and Xu [1999] and many others. Recently, Chen, Li and Ou [2005; 2006] applied the moving plane method to integral equations to obtain the symmetry, monotonicity and nonexistence properties of the solutions to the integral equations. Instead of extensive use of the maximum principle of differential equations, the moving plane method in integral form explores various specific features of the integral equation itself. (See also the work of Li [2004] on the moving sphere method in integral form.) Subsequently, more work has been done in the direction of the moving plane method in integral form: see [Jin and Li 2006; Ma and Chen 2006; 2008; Qing and Raske 2006; Hang 2007; Li and Ma 2008] and others. Nevertheless, all work on the moving plane method in integral form has been on the whole space $\mathbb{R}^n$ (or on a ball in [Chen and Zhu 2011]). In this paper, we will adapt the moving plane method in integral form in a half-space to prove the axial symmetry of nonnegative solutions to a class of integral equations associated to the Dirichlet problem of polyharmonic equations on a half-space.

By virtue of the Hardy–Littlewood–Sobolev inequality or its general form, the weighted Hardy–Littlewood–Sobolev inequality, and comparison of the solution and its reflection with the plane, we can start moving the plane from infinity. Furthermore the plane must be moved to a critical point. As a result, symmetry and monotonicity properties are derived.

We also obtain regularity results for

$$u(x) = \int_{\mathbb{R}^n_+} G(x, y)|u|^{p-1}u\, dy \tag{6}$$

in the case of $n > 2m$ and $p > n/(n-2m)$. The method we use here is called "regularity lifting" based on the contraction mapping theorem. It is an elegant and powerful tool in obtaining regularity of solutions. (See [Chen and Li 2010], and also Section 3 below for more details.)

Theorem 3. *Let $u(x)$ be a solution of* (6). *Assume that $p > n/(n-2m)$ and $u(x) \in L^{(p-1)n/(2m)}(\mathbb{R}^n_+)$. Then u is bounded in $\mathbb{R}^n_+$ and moreover in $L^s(\mathbb{R}^n_+)$ for $s > n/(n-2m)$.*

Next, we consider the nonnegative solutions in (6), that is,

$$u(x) = \int_{\mathbb{R}^n_+} G(x, y)u^p(y)\, dy. \tag{7}$$

Corollary 2. *Assume $p > n/(n-2m)$ and let $u(x)$ be the nonnegative solution in* (7) *and $u(x) \in L^{(p-1)n/(2m)}(\mathbb{R}^n_+)$. Then u is trivial.*

We further study the properties of solutions in (7) under the weaker assumption that $u \in L^{2n/(n-2m)}_{\mathrm{loc}}(\mathbb{R}^n_+)$. We obtain that

Theorem 4. *Let $u(x)$ be the nonnegative solution in* (7) *with $u \in L^{2n/(n-2m)}_{\mathrm{loc}}(\mathbb{R}^n_+)$ and assume $1 < p < (n+2m)/(n-2m)$. Then $u(x)$ only depends on x_n.*

The paper is arranged as follows. In Section 2, we present some properties of Green's function for polyharmonic operators in a half-space. Section 3 is devoted to the proof of Theorem 1 using the method of moving planes in integral form.

In Section 4, we verify Theorem 2 with a similar technique of moving planes in integral form. We establish Theorem 3 by the contraction mapping method in Section 5. Theorem 4 is obtained in Section 6. In this paper C denotes a positive constant, which may vary from line to line.

2. Properties of Green's function

In this section, we introduce some results about the Green's function $G = G(x, y)$ of $(-\Delta)^m$ in $\mathbb{R}^n_+$ corresponding to a Dirichlet boundary condition. For fixed $y \in \mathbb{R}^N_+$,

$$\begin{cases} (-\Delta)^m G(x, y) = \delta(x-y) & \text{in } \mathbb{R}^n_+, \\ G = \dfrac{\partial G}{\partial x_n} = \cdots = \dfrac{\partial^{m-1} G}{\partial x_n^{m-1}} = 0 & \text{on } x_n = 0. \end{cases} \tag{8}$$

Define

$$\begin{aligned} d(x, y) &= |x-y|^2 \quad \text{for } x, y \in \mathbb{R}^n_+, \\ \theta(x, y) &= \begin{cases} x_n y_n & \text{if } x, y \in \mathbb{R}^n_+, \\ 0 & x \notin \mathbb{R}^n_+ \text{ or } y \notin \mathbb{R}^n_+. \end{cases} \end{aligned} \tag{9}$$

Using a rescaling argument from the Green's function of a polyharmonic elliptic equation in the ball [Boggio 1905] (see also [Bachar et al. 2004]), $G = G(x, y)$ has the form

$$\begin{aligned} G(x, y) &= K_n^m |x-y|^{2m-n} \int_0^{4\theta(x,y)/|x-y|^2} \frac{z^{m-1}}{(z+1)^{n/2}}\, dz \\ &= K_n^m H(d(x, y), \theta(x, y)). \end{aligned}$$

Here K_n^m is a positive constant and

$$H : (0,\ \infty) \times [0,\ \infty) \to \mathbb{R}, \quad H(s, t) = s^{m-n/2} \int_0^{4t/s} \frac{z^{m-1}}{(z+1)^{n/2}}\, dz$$

with

$$d(x, y) = s, \quad \theta(x, y) = t.$$

We now introduce some notation which will be used extensively in this paper. Let $x = \{x_1, \dots, x_i, \dots, x_n\}$ for $1 \le i \le n$, let $T^i_\lambda = \{x \mid x_i = \lambda\}$ and let $x^i_\lambda = \{x_1, \dots, 2\lambda - x_i, \dots, x_n\}$ be the reflection of the point x about the plane T^i_λ. Set $\Sigma^i_\lambda = \{x \in \mathbb{R}^n_+ \mid x_i < \lambda\}$. If $1 \le i < n$ then λ can be any real number. For $i = n$, since we will move the plane from $x_n = 0$ to positive infinity, we only consider the case that λ is positive. In this case, we introduce $(\Sigma^n_\lambda)^C = \{x_\lambda \mid x \in \Sigma^n_\lambda\}$.

To simplify the presentation, we will drop the superscript i from T^i_λ, Σ^i_λ, x^i_λ, etc. when $1 \le i < n$ or $i = n$ is given and no confusion is caused.

We will prove the following properties for $G(x, y)$ in a half-space which will be used in next section. See [Berchio et al. 2008; Chen and Zhu 2011] for a similar lemma on Green's function on a ball.

Lemma 1. (i) *Let* $\lambda \in (-\infty,\ 0)$. *For any* $x, y \in \Sigma_\lambda,\ x \neq y$,

$$G(x_\lambda, y_\lambda) > \max\{G(x_\lambda, y), G(x, y_\lambda)\}, \tag{10}$$

$$G(x_\lambda, y_\lambda) - G(x, y) = G(x_\lambda, y) - G(x, y_\lambda) = 0 \quad \text{if } 1 \leq i < n. \tag{11}$$

(ii) *Let* $\lambda \in (0,\ \infty)$. *For any* $x, y \in \Sigma_\lambda,\ x \neq y$,

$$G(x_\lambda, y_\lambda) - G(x, y) > |G(x_\lambda, y) - G(x, y_\lambda)| \quad \text{if } i = n. \tag{12}$$

(iii) *Let* $\lambda \in (0, \infty)$. *For any* $x \in \Sigma_\lambda$, $y \in \mathbb{R}^n_+ \setminus (\Sigma_\lambda \cup \Sigma^C_\lambda)$,

$$G(x, y) < G(x_\lambda, y) \quad \text{if } i = n. \tag{13}$$

Proof. (i) For $x, y \in \Sigma_\lambda$, obviously $d(x_\lambda, y_\lambda) < d(x, y_\lambda)$. Since θ is only dependent on the n-th variable, in the case $1 \leq i < n$,

$$\theta(x_\lambda, y_\lambda) = \theta(x, y_\lambda) = \theta(x_\lambda, y) = \theta(x, y). \tag{14}$$

In the case $i = n$,

$$\theta(x_\lambda, y_\lambda) > \max(\theta(x, y_\lambda), \theta(x_\lambda, y)) \geq \min(\theta(x, y_\lambda), \theta(x_\lambda, y)) > \theta(x, y). \tag{15}$$

We compute

$$\begin{aligned} H(s,t) &= s^{m-n/2} \int_0^{4t/s} \frac{z^{m-1}}{(z+1)^{n/2}}\, dz, \\ &= \int_0^{4t} \frac{z^{m-1}}{(z+s)^{n/2}}\, dz. \end{aligned}$$

For $s, t > 0$,

$$\frac{\partial H}{\partial s} = -\frac{n}{2} \int_0^{4t} \frac{z^{m-1}}{(z+s)^{n/2+1}}\, dz < 0, \tag{16}$$

$$\frac{\partial H}{\partial t} = \frac{4(4t)^{m-1}}{(4t+s)^{n/2}} > 0, \tag{17}$$

$$\frac{\partial^2 H}{\partial t \partial s} = \frac{-2n(4t)^{m-1}}{(t+s)^{n/2+1}} < 0. \tag{18}$$

From (14), (15), (16) and (17), we arrive at (10).

In the case $1 \leq i < n$, since $d(x_\lambda, y_\lambda) = d(x, y)$ and $d(x, y_\lambda) = d(x_\lambda, y)$, and moreover, $\theta(x, y)$ is a function in x_n and y_n, it is easy to verify (11).

(ii) If $i = n$, from (15), (18),

$$\begin{aligned}
G(x_\lambda, y_\lambda) - G(x, y) &= K_n^m \int_{\theta(x,y)}^{\theta(x_\lambda, y_\lambda)} \frac{\partial H(d(x, y), t)}{\partial t}\, dt \\
&> K_n^m \int_{\theta(x,y)}^{\theta(x_\lambda, y_\lambda)} \frac{\partial H(d(x_\lambda, y), t)}{\partial t}\, dt \\
&\geq K_n^m \int_{\min(\theta(x,y_\lambda),\theta(x_\lambda,y))}^{\max(\theta(x,y_\lambda),\theta(x_\lambda,y))} \frac{\partial H(d(x_\lambda, y), t)}{\partial t}\, dt \\
&= K_n^m \left| H(d(x_\lambda, y),\, \theta(x_\lambda, y)) - H(d(x, y_\lambda),\, \theta(x, y_\lambda)) \right| \\
&= |G(x_\lambda, y) - G(x, y_\lambda)|,
\end{aligned}$$

which confirms (12).

(iii) For $i = n$, if $x \in \Sigma_\lambda$ and $y \in \mathbb{R}^n_+ \setminus (\Sigma_\lambda \cup \Sigma_\lambda^C)$, we have

$$d(x, y) > d(x_\lambda, y)$$

and

$$\theta(x, y) < \theta(x_\lambda, y).$$

Then (13) follows immediately from (16) and (17). □

3. Proof of Theorem 1

Let $u^\lambda(x) = u(x_\lambda)$. Once again, we have dropped the superscript i from x_λ^i, etc....

Lemma 2. *The following equality holds*:

$$u(x) - u(x_\lambda) \leq \int_{\Sigma_\lambda} \big(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\big)\big(f(u(y)) - f(u^\lambda(y))\big).$$

Proof. First consider the case $1 \leq i < n$. In this situation,

$$\begin{aligned}
u(x) &= \int_{\Sigma_\lambda} G(x\,, y) f(u(y))\, dy + \int_{\Sigma_\lambda} G(x\,, y_\lambda) f(u^\lambda(y))\, dy, \\
u^\lambda(x) &= \int_{\Sigma_\lambda} G(x_\lambda\,, y) f(u(y))\, dy + \int_{\Sigma_\lambda} G(x_\lambda\,, y_\lambda) f(u^\lambda(y))\, dy.
\end{aligned}$$

Combining this with (11) in Lemma 1, we derive

$$u(x) - u(x_\lambda) = \int_{\Sigma_\lambda} \big(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\big)\big(f(u(y)) - f(u^\lambda(y))\big)\, dy.$$

Now assume $i = n$.

$$u(x) = \int_{\Sigma_\lambda} G(x\,,y) f(u(y))\,dy + \int_{\Sigma_\lambda} G(x\,,y_\lambda) f(u^\lambda(y))\,dy + \int_{\mathbb{R}^n_+ \setminus (\Sigma_\lambda \cup \Sigma_\lambda^C)} G(x,y) f(u(y))\,dy,$$

$$u^\lambda(x) = \int_{\Sigma_\lambda} G(x_\lambda\,,y) f(u(y))\,dy + \int_{\Sigma_\lambda} G(x_\lambda\,,y_\lambda) f(u^\lambda(y))\,dy + \int_{\mathbb{R}^n_+ \setminus (\Sigma_\lambda \cup \Sigma_\lambda^C)} G(x_\lambda,y) f(u(y))\,dy.$$

From (12), (13) in Lemma 1 and the property of f_i, we derive

$$u(x) - u(x_\lambda) \le \int_{\Sigma_\lambda} \big(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\big)\big(f(u(y)) - f(u^\lambda(y))\big)\,dy.$$

This completes the proof of the lemma. □

Lemma 3 (equivalent form of the Hardy–Littlewood–Sobolev inequality). *Assume $0 < \alpha < n$ and $\Omega \subset \mathbb{R}^n$. Let $g \in L^{np/(n+\alpha p)}(\Omega)$ for $n/(n-\alpha) < p < \infty$. Define*

$$Tg(x) = \int_\Omega \frac{1}{|x-y|^{n-\alpha}} g(y)\,dy.$$

Then

$$\|Tg\|_{L^p(\Omega)} \le C(n,p,\alpha)\|g\|_{L^{np/(n+\alpha p)}(\Omega)}. \tag{19}$$

The proof of this lemma is standard and follows from the $L^{np/(n+\alpha p)} \to L^p$ boundedness for the fractional integral operator of order α when $p > n/(n-\alpha)$ (see, for example, [Stein 1970]).

Next we will prove $u(x)$ is axially symmetric. In the proof of nonexistence of solutions, we only need to show that $u(x)$ is increasing in the x_n direction. For the sake of completeness, we present the whole picture in half-space of the moving plane method in integral form. This way of showing axial symmetry is also is used in proof of Theorem 4. We first show that $u(x)$ is radially symmetric with respect to some $x_0 \in \mathbb{R}^{n-1}$ for any fixed x_n. Then we prove that $u(x)$ is nondecreasing in the x_n direction. To prove radial symmetry, there are two steps in carrying out the process of moving planes. In Step 1, we need to show that the plane can be moved near infinity, that is, we will show that $u(x) \le u(x_\lambda)$ for sufficiently negative λ. In Step 2, we prove that the plane has to move to a critical point. By a contradiction argument, radial symmetry is obtained. To prove $u(x)$ is nondecreasing, we only need to carry out Step 1 in the x_n direction.

Lemma 4. *If (f_1) and either (f_2) or $(\tilde{f_2})$ hold, $u \in L^r(\mathbb{R}^n_+)$ for some $r > n/(n-2m)$ is a nonnegative solution of* (1), *then u is axially symmetric with respect to some line parallel to the x_n-axis and $u(x)$ is nondecreasing in the x_n direction.*

Proof. First consider the case $1 \le i < n$. Without loss of generality, let $i = 1$. Start to move the plane in the x_1 coordinate. Proving the symmetry of solutions in the x_1 coordinate and taking the same steps with all coordinates except x_n gives radial symmetry of solutions with respect to some $x_0 \in R^{n-1}$ with fixed x_n.

Step 1: Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid u(x) > u^\lambda(x)\},$$
$$w^\lambda(x) = u(x) - u^\lambda(x).$$

By (10) in Lemma 1, positivity of the Green's function, the properties of f and Lemma 2,

$$\begin{aligned}
u(x) - u(x_\lambda) \le & \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \big(G(x_\lambda, y_\lambda) - G(x, y_\lambda)\big)\big(f(u(y)) - f(u^\lambda(y))\big)\,dy \\
& + \int_{\Sigma_\lambda^-} \big(G(x_\lambda, y_\lambda) - G(x, y_\lambda)\big)\big(f(u(y)) - f(u^\lambda(y))\big)\,dy \\
\le & \int_{\Sigma_\lambda^-} G(x_\lambda, y_\lambda)\big(f(u(y)) - f(u^\lambda(y))\big)\,dy \\
\le & \, C \int_{\Sigma_\lambda^-} |x-y|^{2m-n} \int_0^{4x_n y_n/|x-y|^2} \frac{z^{m-1}}{(z+1)^{n/2}}\,dz\big(f(u(y)) - f(u^\lambda(y))\big)\,dy \\
\le & \, C \int_{\Sigma_\lambda^-} |x-y|^{2m-n}\big(f(u(y)) - f(u^\lambda(y))\big)\,dy.
\end{aligned}$$

By the Hardy–Littlewood–Sobolev inequality, the mean value theorem, and the assumption that $u \in L^r(\mathbb{R}^n_+)$ for some $r > n/(n-2m)$, we deduce

$$\|w^\lambda\|_{L^r(\Sigma_\lambda^-)} \le C \left\| \frac{\partial f(\theta u + (1-\theta)u^\lambda)}{\partial u} w^\lambda \right\|_{L^{nr/(n+2mr)}(\Sigma_\lambda^-)},$$

where $0 < \theta < 1$. Furthermore, by Hölder's inequality, we get

$$\|w^\lambda\|_{L^r(\Sigma_\lambda^-)} \le C \left\| \frac{\partial f(\theta u + (1-\theta)u^\lambda)}{\partial u} \right\|_{L^{n/(2m)}(\Sigma_\lambda^-)} \|w^\lambda\|_{L^r(\Sigma_\lambda^-)}. \tag{20}$$

From property (f_2) of the function f, if λ is sufficiently negative,

$$C \left\| \frac{\partial f(\theta u + (1-\theta)u^\lambda)}{\partial u} \right\|_{L^{n/(2m)}(\Sigma_\lambda^-)} \le \tfrac{1}{2},$$

then

$$\|w^\lambda\|_{L^r(\Sigma_\lambda^-)} \le \tfrac{1}{2} \|w^\lambda\|_{L^r(\Sigma_\lambda^-)},$$

which implies that $\|w^\lambda\|_{L^r(\Sigma_\lambda^-)} = 0$. Therefore, Σ_λ^- must be of measure zero. Thus, $u(x) \le u^\lambda(x)$ a.e. in Σ_λ for sufficiently negative λ.

Step 2: Continue to move the plane $x_1 = \lambda$ to the right as long as Step 1 holds. We claim that there exists some critical point

$$\lambda_0 = \sup_\lambda \{u^\lambda(x) \ge u(x) \mid -\infty < \lambda < 0,\ x \in \Sigma_\lambda\} \tag{21}$$

such that $w^{\lambda_0} \equiv 0$.

If $\lambda_0 = 0$ in Step 1, then $u(x) \le u(x_{\lambda_0})$ in Σ_{λ_0}. Move the plane from positive infinity to the origin and argue in the same way as in Step 1. If $\lambda_0 = 0$ again, it is obvious that $u(x) \equiv u(x_{\lambda_0})$. Hence, $w^{\lambda_0}(x) \equiv 0$.

If $\lambda_0 < 0$, we claim that

$$w^{\lambda_0}(x) < 0 \quad \text{or} \quad w^{\lambda_0}(x) \equiv 0 \quad \text{for } x \in \Sigma_{\lambda_0}.$$

Suppose that there exists some x_0 in Σ_{λ_0} such that $w^{\lambda_0}(x_0) = 0$, but $w^{\lambda_0} \not\equiv 0$. By (11) in Lemma 1,

$$u(x_0) - u_{\lambda_0}(x_0) = \int_{\Sigma_{\lambda_0}} \big(G(x_{\lambda_0}, y_{\lambda_0}) - G(x, y_{\lambda_0})\big)\big(f(u(y)) - f(u_{\lambda_0}(y))\big)\, dy.$$

Moreover, by (10), $f(u(y)) \equiv f(u(y_{\lambda_0}))$. This contradicts with the fact that the function f satisfies (f_1), which verifies the claim.

Next, we show the plane can be moved to the right a little bit farther if $w^{\lambda_0}(x) < 0$. By the assumption (f_2) on the function f, for any small ϵ, there exists a large enough ball $\mathbb{B}_R(0)$ such that

$$\left\| \frac{\partial f(\theta u + (1-\theta)u_\lambda)}{\partial u} \right\|_{L^{n/(2m)}(\mathbb{R}^n_+ \setminus \mathbb{B}_R)} < \epsilon.$$

From Lusin's theorem, for any δ, there exists a closed set F_δ such that $w_{\lambda_0}|_{F_\delta}$ is continuous, with $F_\delta \subset E := \mathbb{B}_R(0) \cap \Sigma_{\lambda_0}$ and $m(E - F_\delta) < \delta$. As $w^{\lambda_0}(x) < 0$ in the interior of Σ_{λ_0}, $w^{\lambda_0}(x) < 0$ in F_δ.

Choosing ϵ_1 sufficiently small, for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon_1)$, we have

$$w^\lambda(x) < 0 \quad \text{for all } x \in F_\delta$$

by continuity. It follows that for such λ,

$$\Sigma_\lambda^- \subset M := (\mathbb{R}^n_+ \setminus \mathbb{B}_R(0)) \cup (E \setminus F_\delta) \cup \big((\Sigma_\lambda \setminus \Sigma_{\lambda_0}^-) \cap \mathbb{B}_R(0)\big).$$

Choosing ϵ, δ and ϵ_1 small enough and using absolute continuity of integration, we derive

$$C \left\| \frac{\partial f(u_\lambda + \theta w^\lambda)}{\partial u} \right\|_{L^{n/(2m)}(M)} \le \tfrac{1}{2}.$$

Consequently from (20), $\|w^\lambda(x)\|_{L^q(\Sigma_\lambda^-)} = 0$. Then, Σ_λ^- must be of measure zero, which contradicts the definition of λ_0. Hence, $w^{\lambda_0} \equiv 0$.

This completes Steps 1 and 2 for $1 \le i < n$.

For the case $i = n$, start moving the plane from $x_n = 0$ as in the case $1 \le i < n$. Choosing $\lambda > 0$ sufficiently small, Step 1 is carried out similarly, which implies that $w^\lambda(x) \le 0$. Next we prove that if $w^\lambda(x) \not\equiv 0$,

$$\lambda_0 = \sup_\lambda \{u^\lambda(x) > u(x),\ \lambda > 0,\ x \in \Sigma_\lambda\} = \infty.$$

If not, then $\lambda_0 < \infty$. It is known that $w^{\lambda_0}(x) < 0$ or $w^{\lambda_0}(x) \equiv 0$ for any $x \in \Sigma_{\lambda_0}$. Hence, $w^\lambda(x) < 0$ for any λ. If $w^\lambda(x) \equiv 0$, since $u(x', x_n) = 0$ on $x_n = 0$, then $u(x) \equiv 0$. Therefore $u(x)$ is nondecreasing in the x_n direction. Thus we have completed the proof of Lemma 4. □

Proof of Theorem 1. Since $u(x)$ is nondecreasing in the x_n direction by Lemma 4 and $u \in L^r(\mathbb{R}^n_+)$ for some $r > n/(n-2m)$ by assumption, then for any $a \in \mathbb{R}_+$,

$$\int_{\mathbb{R}^n_+} |u(x', x_n)|^r \, dx' \, dx_n \ge \int_{\mathbb{R}^{n-1}} \int_a^\infty |u(x', a)|^r \, dx' \, dx_n.$$

The integrability of nonnegative u implies $u(x', a) = 0$ for any a and $x' \in \mathbb{R}^{n-1}$. Hence $u(x) = 0$ in $\mathbb{R}^n_+$. □

4. Proof of Theorem 2

Lemma 5. *If* (f_i^1) *and either* (f_i^2) *or* $(\tilde{f}_i^2)$ *hold and* $u_i \in L^r(\mathbb{R}^n_+)$ *for some* $r > n/(n-2m)$ *is the nonnegative solution of* (4), *then* u_i *is axially symmetric with respect to some line parallel to the* x_n*-axis and* $u_i(x)$ *is nondecreasing in the* x_n *direction for* $i = 1, \dots, N$.

Proof. Step 1: We proceed with the same process as in the proof of Theorem 1. First consider all coordinates except x_n. Without loss of generality, move the plane in the x_1 coordinate. Let

$$\Sigma_\lambda^j = \{x \in \Sigma_\lambda \mid u_j(x) > u_j(x_\lambda)\},$$
$$w_i^\lambda(x) = u_i(x) - u_i(x_\lambda).$$

The same technique as in Lemma 2 and properties (f_i^1) of the functions f_i give

$$\begin{aligned} w_i^\lambda(x) &\\ &\le \int_{\Sigma_\lambda} \big(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\big)\big(f_i(y', y_n, u_1, \dots, u_N) - f_i(y', y_n, u_1^\lambda, \dots, u_N^\lambda)\big) \\ &= \sum_{j=1}^N \int_{\Sigma_\lambda} \big(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\big) K_{i,j}(y, \lambda)\, dy \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \int_{\Sigma_\lambda \setminus \Sigma_\lambda^j} \bigl(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\bigr) K_{i,j}(y,\lambda)\, dy \\
&\qquad\qquad + \sum_{j=1}^{N} \int_{\Sigma_\lambda^j} \bigl(G(x_\lambda,\ y_\lambda) - G(x,\ y_\lambda)\bigr) K_{i,j}(y,\lambda)\, dy \\
&\le \sum_{j=1}^{N} \int_{\Sigma_\lambda^j} G(x_\lambda,\ y_\lambda) K_{i,j}(y,\lambda)\, dy,
\end{aligned}
$$

where

$$
\begin{aligned}
&K_{i,j}(y,\lambda) \\
&\quad = f_i(y', y_n, u_1^\lambda, \dots, u_{j-1}^\lambda, u_j, \dots, u_N) - f_i(y', y_n, u_1^\lambda, \dots, u_{j-1}^\lambda, u_j^\lambda, \dots, u_N).
\end{aligned}
$$

Estimating in the same way as in the proof of Lemma 4,

$$
u_i(x) - u_i(x_\lambda) \le C \sum_{j=1}^{N} \int_{\Sigma_\lambda^j} |x-y|^{2m-n} K_{i,j}(y,\lambda)\, dy.
$$

By the Hardy–Littlewood–Sobolev inequality, the mean value theorem and the assumption that $u \in L^r(\mathbb{R}^n_+)$ for some $r > n/(n-2m)$, we deduce

$$
\|w_i^\lambda\|_{L^r(\Sigma_\lambda^i)} \le C \sum_{j=1}^{N} \left\| \frac{\partial f_i(y', y_n, u_1^\lambda, \dots, u_j^\lambda + \theta_j w_j^\lambda, \dots, u_N)}{\partial u_j} w_j^\lambda \right\|_{L^{nr/(n+2mr)}(\Sigma_\lambda^j)},
$$

where $0 < \theta_j < 1$. Then, from Hölder's inequality, we have

$$
\|w_i^\lambda\|_{L^r(\Sigma_\lambda^i)} \le C \sum_{j=1}^{N} \left\| \frac{\partial f_i(y', y_n, u_1^\lambda, \dots, u_j^\lambda + \theta_j w_j^\lambda, \dots, u_N)}{\partial u_j} \right\|_{L^{n/(2m)}(\Sigma_\lambda^j)} \|w_j^\lambda\|_{L^r(\Sigma_\lambda^j)}.
$$

Moreover, taking the sum from i to N gives

$$
\sum_{i=1}^{N} \|w_i^\lambda\|_{L^r(\Sigma_\lambda^i)} \le C \sum_{i=1}^{N} \sum_{j=1}^{N} \left\| \frac{\partial f_i}{\partial u_j} \right\|_{L^{n/(2m)}(\Sigma_\lambda^j)} \|w_j^\lambda\|_{L^r(\Sigma_\lambda^j)}. \tag{22}
$$

By virtue of the properties (f_i^1) and (f_i^2) of the functions f_i, we can choose λ negative enough that

$$
C \sum_{i=1}^{N} \sum_{j=1}^{N} \left\| \frac{\partial f_i}{\partial u_j} \right\|_{L^{n/(2m)}(\Sigma_\lambda^j)} \le \tfrac{1}{2}.
$$

This implies that

$$
\sum_{i=1}^{N} \|w_i^\lambda\|_{L^r} = 0.
$$

Therefore, Σ_i^λ must be of measure zero and $u_i(x) \le u_i^\lambda(x)$ a.e. in Σ_λ for every $1 \le i < n$.

Step 2: We will prove that the plane can be moved to a critical point

$$\lambda_0 = \sup\{\lambda \mid u_i(x) \le u_i^\lambda(x),\ -\infty < \lambda < 0,\ x \in \Sigma_\lambda,\ i = 1, \dots, N\}$$

such that $w_i^{\lambda_0} \equiv 0$.

If $\lambda_0 = 0$, move the plane from the positive infinity to the origin and argue as before. Hence assume $\lambda_0 < 0$, so obviously $u_i(x) \le u_i^{\lambda_0}(x)$. We claim that

$$u_i(x) < u_i^{\lambda_0}(x) \quad \text{or} \quad u_i(x) \equiv u^{\lambda_0}(x)$$

in Σ_{λ_0} for $i = 1, \dots, N$. If not, then there exists some $i \in \{i_1, \dots, i_{l_0}\} \subset \{1, \dots, N\}$ such that $u_i(x^0) = u_i^{\lambda_0}(x^0)$ for some $x^0 \in \Sigma_{\lambda_0}$, but $u_i(x) \not\equiv u_i^{\lambda_0}(x)$. Therefore $u_i(x) < u_i^{\lambda_0}(x)$ for $i \in S = \{1, \dots, N\} \setminus \{i_1, \dots, i_{l0}\}$ for any $x \in \Sigma_{\lambda_0}$. Now

$$\begin{aligned} &w_i^\lambda(x^0) \\ &\le \int_{\Sigma_\lambda} \big(G(x_\lambda^0,\ y_\lambda) - G(x^0,\ y_\lambda)\big)\big(f_i(y', y_n, u_1, \dots, u_N) - f_i(y', y_n, u_1^\lambda, \dots, u_N^\lambda)\big) \\ &\le 0, \end{aligned}$$

thus, $f_i(y', y_n, u_1, \dots, u_N) \equiv f_i(y', y_n, u_1^\lambda, \dots, u_N^\lambda)$ for $i \in \{i_1, \dots, i_{l0}\}$. Hence

$$\sum_{i=i_1}^{i_{l_0}} f_i(y', y_n, u_1, \dots, u_N) \equiv \sum_{i=i_1}^{i_{l_0}} f_i(y', y_n, u_1^\lambda, \dots, u_N^\lambda).$$

This contradicts the assumption that f_i are essentially related, which implies the claim is true.

Suppose $u_i(x) < u_i^{\lambda_0}(x)$ in Σ_{λ_0} for $i = 1, \dots, N$. In the spirit of Lemma 4, we will show that the plane can move to the right a little bit further. By the assumption (f_i^2), for any small ϵ, there exists a large enough ball $\mathbb{B}_R(0)$ such that

$$\left\| \frac{\partial f_i}{\partial u_j} \right\|_{L^{n/(2m)}(\mathbb{R}_+^n \setminus \mathbb{B}_R)} < \epsilon,$$

for $i, j = 1, \dots, N$. From Lusin's theorem, for any δ, there exists a closed set F_δ such that $w_i^{\lambda_0}|_{F_\delta}$ is continuous for $i = 1, \dots, N$, with $F_\delta \subset E := \mathbb{B}_R(0) \cap \Sigma_{\lambda_0}$ and $m(E - F_\delta) < \delta$. As $w_i^{\lambda_0}(x) < 0$ in the interior of Σ_{λ_0}, $w^{\lambda_0}(x) < 0$ in F_δ. Choosing ϵ_1 sufficiently small gives that for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon_1)$,

$$w_i^\lambda < 0 \quad \text{for all } x \in F_\delta$$

by continuity. It follows that for such λ,

$$\Sigma_\lambda^i \subset M := (\mathbb{R}_+^n \setminus \mathbb{B}_R(0)) \cup (E \setminus F_\delta) \cup \big((\Sigma_\lambda \setminus \Sigma_{\lambda_0}^-) \cap \mathbb{B}_R(0)\big)$$

for $i=1,\dots,N$. Choosing ϵ, δ and ϵ_1 small enough and using absolute continuity of integration, we conclude that

$$C\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|\frac{\partial f_i}{\partial u_j}\right\|_{L^{n/(2m)}(M)}\le\tfrac{1}{2}.$$

Then, similarly to (22), $\sum_{i=1}^{N}\|w_i^\lambda\|_{L^q}=0$. Therefore, Σ_λ^i must be of measure zero, which contradicts the definition of λ_0. Hence $w_i^{\lambda_0}\equiv 0$ in Σ_{λ_0} for $i=1,\dots,N$.

For the x_n-coordinate, start moving the plane from $x_n=0$ to positive infinity. If $\lambda>0$ is sufficiently small, then we can show as in Step 1 that $w_i^\lambda(x)<0$.

Next, we prove that, if $w_i^\lambda(x)\not\equiv 0$, then

$$\lambda_0=\sup_\lambda\{u_i^\lambda(x)>u(x),\ \lambda>0,\ x\in\Sigma_\lambda,\ i=1,\dots,N\}=\infty.$$

If not, then $\lambda_0<\infty$. It is also known that $w_i^{\lambda_0}<0$ or $w_i^{\lambda_0}\equiv 0$ for $x\in\Sigma_{\lambda_0}$ for $i=1,\dots,N$. Hence $w_i^\lambda<0$ for any λ. If $w_i^\lambda(x)\equiv 0$, since $u_i(x',x_n)=0$ on $x_n=0$, then $u_i(x)\equiv 0$. Therefore, $u_i(x)$ is nondecreasing in the x_n direction for every i. □

Proof of Theorem 2. By Lemma 5, $u_i(x)$ is nondecreasing in the x_n direction. By the assumption that $u\in L^r(\mathbb{R}^n_+)$ for some $r>n/(n-2m)$, for any $a\in\mathbb{R}_+$, then

$$\int_{\mathbb{R}^n_+}|u_i(x',x_n)|^r\,dx'\,dx_n\ge\int_{\mathbb{R}^{n-1}}\int_a^\infty|u_i(x',a)|^r\,dx'\,dx_n.$$

The integrability of nonnegative u_i implies $u_i(x',a)=0$ for any a and $x'\in\mathbb{R}^{n-1}$. Thus $u_i(x)=0$ in $\mathbb{R}^n_+$ for $i=1,\dots,N$. □

5. Proof of Theorem 3

In this section, we prove the regularity of the solutions in (6) which is related to the Dirichlet problems of polyharmonic elliptic equations. For the convenience of the reader, we present a regularity lifting lemma (Lemma 6), initially used in [Chen and Li 2010].

Let Z be a given vector space. Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be two norms on Z. Define a new norm $\|\cdot\|_Z$ by

$$\|\cdot\|_Z=\sqrt[p]{\|\cdot\|_X^p+\|\cdot\|_Y^p}.$$

For simplicity, we assume that Z is complete with respect to the norm $\|\cdot\|_Z$. Let X and Y be the completions of Z under $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Here p can be chosen between 1 and ∞, according to need. It is easy to see that $Z=X\cap Y$.

Lemma 6 (regularity lifting lemma). *Let T be a contraction map from X into itself and from Y into itself. Assume that $f\in X$, and that there exists a function $g\in Z$ such that $f=Tf+g$. Then f also belongs to Z.*

Proof of Theorem 3. From the integral representation of $G(x, y)$, we have the estimates

$$|G(x, y)| \leq C|x-y|^{2m-n}. \tag{23}$$

Define the linear operator

$$T_u w(x) = \int_{\mathbb{R}^n_+} G(x, y)|u|^{p-1} w\, dy.$$

For any real number $a > 0$, define

$$\begin{cases} u_a(x) = u(x) & \text{if } |u(x)| > a \text{ or } |x| > a, \\ u_a(x) = 0 & \text{otherwise.} \end{cases} \tag{24}$$

Let $u_b(x) = u(x) - u_a(x)$. Since u satisfies (6), it follows that

$$u_a(x) = \int_{\mathbb{R}^n_+} G(x, y)|u_a|^{p-1} u_a\, dy + I(x), \tag{25}$$

where

$$I(x) = \int_{\mathbb{R}^n_+} G(x, y)|u_b|^{p-1} u_b\, dy - u_b(x). \tag{26}$$

We claim that for any $r > n/(n-2m)$

$$I(x) \in L^\infty(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+).$$

In order to prove this claim, by the definition of u_b, we only need to show that

$$B(x) := \int_{\mathbb{R}^n_+} G(x, y)|u_b|^{p-1} u_b\, dy \in L^\infty(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+).$$

By (23),

$$|B(x)| \leq C \int_{\mathbb{R}^n_+} |x-y|^{2m-n} |u_b|^p\, dy. \tag{27}$$

Applying the Hardy–Littlewood–Sobolev inequality to (27) gives that for any $r > n/(n-2m)$,

$$\|B(x)\|_{L^r} \leq C\|u_b\|_{L^{nr/(n+2mr)}} < \infty \tag{28}$$

by the definition of u_b.

For any $x \in \mathbb{B}_{2a}$, we estimate that

$$|B(x)| \leq C \int_{\mathbb{R}^n_+ \cap \mathbb{B}_a} |x-y|^{2m-n}\, dy < \infty, \tag{29}$$

while for any $x \notin B_{2a}$,

$$|B(x)| \le C \int_{\mathbb{R}^n_+ \cap \mathbb{B}_a} |x-y|^{2m-n}\, dy \le C a^{2m-n} \int_{\mathbb{R}^n_+ \cap \mathbb{B}_a} dy < \infty.$$

Combining this inequality with (29) and (28), we conclude that

$$B(x) \in L^\infty(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+).$$

Therefore, we have proved the claim.

Next, we prove T_{u_a} is a contraction on $L^s(\mathbb{R}^n_+)$ for any $s > n/(n-2m)$. By (23), for any $s > n/(n-2m)$,

$$\|T_{u_a} w\|_{L^s(\mathbb{R}^n_+)} \le C \left\| \int_{\mathbb{R}^n_+} |x-y|^{2m-n} |u_a(y)|^{p-1} w(y)\, dy \right\|_{L^s(\mathbb{R}^n_+)}.$$

Using the Hardy–Littlewood–Sobolev inequality, then Hölder's inequality gives

$$\|T_{u_a} w\|_{L^s(\mathbb{R}^n_+)} \le C \| |u_a|^{(p-1)} \|_{L^{n/(2m)}(\mathbb{R}^n_+)} \|w\|_{L^s(\mathbb{R}^n_+)}.$$

Since $u(x) \in L^{(p-1)n/(2m)}(\mathbb{R}^n_+)$, choosing a large enough, we have

$$\|T_{u_a} w\|_{L^s(\mathbb{R}^n_+)} \le \tfrac{1}{2} \|w\|_{L^s(\mathbb{R}^n_+)}. \tag{30}$$

Therefore T_{u_a} is a contraction map on $L^s(\mathbb{R}^n_+)$ for any $s > n/(n-2m)$ when a is sufficiently large. Applying (30) to the case of $s = q = (p-1)n/(2m)$ which is greater than $n/(n-2m)$ when $p > n/(n-2m)$ and to the case of $s > n/(n-2m)$, the regularity lifting lemma implies that the unique solution u_a is in $L^q \cap L^s$, which means $u \in L^q \cap L^s$ for any $s > n/(n-2m)$.

Finally, we claim that $u \in L^\infty(\mathbb{R}^n_+)$.

As in (25) and the definition of u_a, it suffices to prove that

$$A(x) := \int_{\mathbb{R}^n_+} G(x,y) |u_a|^{p-1} u_a\, dy \in L^\infty.$$

For any $x \in \mathbb{R}^n_+$, by (23),

$$|A(x)| \le C \int_{\mathbb{R}^n_+ \cap \mathbb{B}_a(x)} |x-y|^{2m-n} |u_a|^p\, dy + C \int_{\mathbb{R}^n_+ \setminus \mathbb{B}_a(x)} |x-y|^{2m-n} |u_a|^p\, dy.$$

Using Hölder's inequality and the property that $u \in L^q \cap L^s$ for any $s > n/(n-2m)$, respectively, we obtain, for any fixed a,

$$\int_{\mathbb{R}^n_+ \cap \mathbb{B}_{2a}(x)} |x-y|^{2m-n} |u_a|^p\, dy < \infty,$$

$$\int_{\mathbb{R}^n_+ \setminus \mathbb{B}_{2a}(x)} |x-y|^{2m-n} |u_a|^p\, dy < \infty.$$

These estimates imply that $A(x) < \infty$, therefore $u \in L^\infty(\mathbb{R}^n_+)$, so Theorem 3 holds. (Actually, we have proved that $L^s(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ for any $s > n/(n-2m)$.) $\square$

Proof of Corollary 2. From the proof of Theorem 3, we have that $u(x) \in L^q \cap L^{s_0}$ for any $s_0 > n/(n-2m)$, where $q = (p-1)n/(2m)$. Then $u \in L^{2n/(n-2m)}$. Let $f(u) = u^p$ in (1). Obviously the assumptions of f in Theorem 1 are satisfied. Therefore, u is trivial in (7). $\square$

6. Proof of Theorem 4

In this section, we sketch the proof of Theorem 4. For a complete proof, refer to the proofs of Theorems 1 and 2. Let

$$v(x) = \frac{1}{|x|^{n-2m}}\, u\Big(\frac{x}{|x|^2}\Big)$$

be the Kelvin transform of $u(x)$ centered at the origin. Then $v(x)$ solves

$$v(x) = \int_{\mathbb{R}^n_+} G(x, y)|y|^{(n-2m)p-(n+2m)} v^p(y)\, dy. \tag{31}$$

Since $u \in L^{2n/(n-2m)}_{\mathrm{loc}}(\mathbb{R}^n_+)$, then $v \in L^{2n/(n-2m)}(\Omega')$, where Ω' is an arbitrary domain in $\mathbb{R}^n_+$ with $\operatorname{dist}(\Omega', 0) > d > 0$ for some positive d. Moreover, $v \in L^{(p-1)n/(2m)}(\Omega')$, since $1 < p < (n+2m)/(n-2m)$. As in the proofs of Lemma 2 and Theorems 1 and 2, for $1 \le i < n$,

$$\begin{aligned}
&v(x) - v(x_\lambda)\\
&= \int_{\Sigma_\lambda} \big(G(x_\lambda, y_\lambda) - G(x, y_\lambda)\big)\big(|y|^{(n-2m)p-(n+2m)} v^p - |y^\lambda|^{(n-2m)p-(n+2m)} v^p_\lambda\big)\, dy\\
&\le \int_{\Sigma_\lambda} \big(G(x_\lambda, y_\lambda) - G(x, y_\lambda)\big)|y|^{(n-2m)p-(n+2m)}(v^p - v^p_\lambda)\, dy\\
&\le C|\lambda|^{(n-2m)p-(n+2m)} \int_{\Sigma^-_\lambda} G(x_\lambda, y_\lambda)(v^p - v^p_\lambda)\, dy.
\end{aligned}$$

Following the steps in proving the axial symmetry of u in Theorems 1 and 2, we can show that $v(x)$ is axially symmetric with respect to x_n axis. Let (x^1, x_n) and (x^2, x_n) be two points in $\mathbb{R}^n_+$, where x^1, x^2 are arbitrary in $\mathbb{R}^{n-1}$. Let x^0 be the midpoint of the line segment $\overline{x^1x^2}$. Consider the Kelvin transform of $u(x)$ centered at $x^* = (x^0, 0)$, that is,

$$v(x) = \frac{1}{|x-x^*|^{n-2m}} u\Big(\frac{x-x^*}{|x-x^*|^2}\Big).$$

Then $v(x)$ is axially symmetric with respect to $x' = x^0$. In particular, $u(x^1, x_n) = u(x^2, x_n)$. Since x^1 and x^2 are any two points in $\mathbb{R}^{n-1}$, the function $u(x', x_n)$ is constant for any fixed x_n. Therefore, $u(x)$ only depends on x_n.

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