The Maximum Principles and Symmetry results for Viscosity Solutions of Fully Nonlinear Equations

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Abstract. This paper is concerned about maximum principles and radial symmetry for viscosity solutions of fully nonlinear partial differential equations. We obtain the radial symmetry and monotonicity properties for nonnegative viscosity solutions of

$$F(D^2u) + u^p = 0 \qquad \text{in } \mathbb{R}^n \tag{0.1}$$

under the asymptotic decay rate $u = o(|x|^{-\frac{2}{p-1}})$ at infinity, where $p > 1$. As a consequence of our symmetry results, we obtain the nonexistence of any nontrivial and nonnegative solution when F is the Pucci extremal operators. Our symmetry and monotonicity results also apply to Hamilton-Jacobi-Bellman or Isaccs equations. A new maximum principle for viscosity solutions to fully nonlinear elliptic equations is established. As a result, different forms of maximum principles on bounded and unbounded domains are obtained. Radial symmetry, monotonicity and the corresponding maximum principle for fully nonlinear elliptic equations in a punctured ball are shown. We also investigate the radial symmetry for viscosity solutions of fully nonlinear parabolic partial differential equations.

1. Introduction

In studying partial differential equations, it is often of interest to know if the solutions are radially symmetric. In this article, we consider radial symmetry results for viscosity solutions of the fully nonlinear elliptic equations

$$F(D^2u) + u^p = 0 \qquad \text{in } \mathbb{R}^n \tag{1.1}$$

and the Dirichlet boundary value problem in a punctured ball

$$\begin{cases} F(Du, D^2u) + f(u) = 0 & \text{in } \mathbb{B}\backslash\{0\}, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.2}$$

We also obtain the radial symmetry for viscosity solutions of the fully nonlinear parabolic equation

$$\begin{cases} \partial_t u - F(Du, D^2u) - f(u) = 0 & \text{in } \mathbb{R}^n \times (0,\ T], \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.3}$$

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We assume in the above that $F(Du, D^2u)$ is a continuous function defined on $\mathbb{R}^n \times S_n(\mathbb{R})$, where $S_n(\mathbb{R})$ is the space of real, $n \times n$ symmetric matrix, $f(u)$ is a locally Lipschitz continuous function and the initial value $u_0(x)$ is continuous. More precisely, we consider $F : \mathbb{R}^n \times S_n(\mathbb{R}) \to \mathbb{R}$ satisfies the following structure hypothesis.

(F1): There exist $\gamma \geq 0$ and $0 < \Lambda_1 \leq \Lambda_2 < \infty$ such that for all $M, N \in S_n(\mathbb{R})$ and $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(M) - \gamma|\xi_1 - \xi_2| \leq F(\xi_1, M+N) - F(\xi_2, N) \leq \mathcal{M}^+_{\Lambda_1,\Lambda_2}(M) + \gamma|\xi_1 - \xi_2|, \tag{1.4}$$

where $\mathcal{M}^{\pm}_{\Lambda_1,\Lambda_2}$ are the Pucci extremal operators, defined as

$$\mathcal{M}^+_{\Lambda_1,\Lambda_2}(M) = \Lambda_2 \Sigma_{e_i>0} e_i + \Lambda_1 \Sigma_{e_i<0} e_i, \tag{1.5}$$

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(M) = \Lambda_1 \Sigma_{e_i>0} e_i + \Lambda_2 \Sigma_{e_i<0} e_i \tag{1.6}$$

where e_i, $i = 1, \cdots, n$, is an eigenvalue of M.

For any $M = (m_{ij}) \in S_n(\mathbb{R})$, let $M^{(k)}$ be the matrix obtained from M by replacing m_{ik} and m_{kj} by $-m_{ik}$ and $-m_{kj}$ for $i \neq k$, $j \neq k$, respectively. For any vector p, let

$$p^{(k)} = (p_1, \cdots, p_{k-1}, -p_k, p_{k+1}, \cdots, p_n).$$

We assume the following hypothesis for F,

(F2):

$$F(p^{(k)}, M^{(k)}) = F(p, M) \tag{1.7}$$

for $k = 1, \cdots, n$.

Note that M and $M^{(k)}$ have the same eigenvalues. In this sense,

$$\mathcal{M}^{\pm}_{\Lambda_1,\Lambda_2}(M^{(k)}) = \mathcal{M}^{\pm}_{\Lambda_1,\Lambda_2}(M).$$

Under the hypotheses $(F1)$ and $(F2)$, it is nature to see that the following hypotheses hold for the $F(D^2u)$ in (1.1), that is,

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(M) \leq F(M+N) - F(N) \leq \mathcal{M}^+_{\Lambda_1,\Lambda_2}(M), \tag{1.8}$$

$$F(M^{(k)}) = F(M). \tag{1.9}$$

Let $\xi_1 = \xi_2$, the hypothesis $(F1)$ implies that the uniform ellipticity for the fully nonlinear equation. Namely, there exist $0 < \Lambda_1 \leq \Lambda_2 < \infty$ such that

$$\Lambda_1 tr(N) \leq F(\xi_1, M+N) - F(\xi_1, M) \leq \Lambda_2 tr(N)$$

for all $M, N \in S_n(\mathbb{R})$, $N \geq 0$, where $tr(N)$ is the trace of the matrix N. It is easy to see that the Pucci's operators (1.5), (1.6) are extremal in the sense that

$$\mathcal{M}^+_{\Lambda_1,\Lambda_2}(M) = \sup_{A \in \mathcal{A}_{\Lambda_1,\Lambda_2}} tr(AM),$$

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(M) = \inf_{A \in \mathcal{A}_{\Lambda_1,\Lambda_2}} tr(AM),$$

where $\mathcal{A}_{\Lambda_1,\Lambda_2}$ denotes the set of all symmetric matrix whose eigenvalues lie in the interval $[\Lambda_1, \Lambda_2]$.

The moving plane method is a powerful tool to show the radial symmetry of solutions in partial differential equations. This method goes back to A.D. Alexandroff and then Serrin [**21**] applies it to elliptic equations for overdetermined problems. Gidas, Ni and

Nirenberg [**14**] further exploit this tool to obtain radial symmetry of positive C^2 solutions of the Dirichlet boundary problem for

$$\triangle u + f(u) = 0, \qquad f \in C^{0,1}(\mathbb{R})$$

in a ball. Notice that the Laplace operator corresponds to $\Lambda_1 = \Lambda_2 = 1$ in our $F(Du, D^2u)$. In [**15**], Gidas, Ni and Nirenberg extend their techniques to elliptic equations in $\mathbb{R}^n$. By assuming that the solutions decay to zero at infinity at a certain rate, the radial symmetry of positive classical solutions is also derived. Further extensions and simpler proofs are due to Berestycki and Nirenberg [**2**] and C. Li [**16**]. For the detailed account and applications of the moving plane method for semilinear elliptic equations, we refer to Chen and Li's book [**3**] and references therein.

Radial symmetry results for classical solutions of fully nonlinear elliptic equations are considered [**16**] and [**17**]. Recently, Da Lio and Sirakov [**12**] studied the radial symmetry for viscosity solutions of fully nonlinear elliptic equations. The moving plane method is adapted to work in the setting of viscosity solutions. We would like to mention that, in these quoted results for radial symmetry in $\mathbb{R}^n$, a supplementary hypothesis that $f(u)$ is nonincreasing in a right neighborhood of zero is required. In the context of fully nonlinear equation $F(x, u, Du, D^2u) = 0$, it is equivalent to say that the operator F is proper in a right neighborhood of zero, i.e. the operator F is nonincreasing in u in the case that u is small.

We are particularly interested in the nonnegative viscosity solutions of

$$F(D^2u) + u^p = 0 \qquad \text{in } \mathbb{R}^n \tag{1.10}$$

for $p > 1$. Note that the proper assumption (that is, nonincreasing in u) for fully nonlinear equation in (1.10) is violated since $f(u) = u^p$ is not nonincreasing any more. So the previous results no longer hold for (1.10). The typical models of (1.10) are the equations

$$\mathcal{M}^{\pm}_{\Lambda_1,\Lambda_2}(D^2u) + u^p = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.11}$$

It is well known that the moving plane method and Kelvin transform provide an elegant way of obtaining the Liouville-type theorems (i.e. the nonexistence of any solution) in [**4**]. For (1.11), the critical exponent for nonexistence of any viscosity solution is still an open problem, since the Kelvin transform does not seem to be available. Curti and Lenoi [**5**] consider the nonnegative supersolutions of (1.11), that is,

$$\mathcal{M}^{\pm}_{\Lambda_1,\Lambda_2}(M) + u^p \le 0 \qquad \text{in } \mathbb{R}^n. \tag{1.12}$$

They show that the inequality (1.12) with $\mathcal{M}^{+}_{\Lambda_1,\Lambda_2}$ has no non-trivial solution for $1 < p \le \frac{n^*}{n^*-2}$, the inequality (1.12) with $\mathcal{M}^{-}_{\Lambda_1,\Lambda_2}$ has no non-trivial solution provided $1 < p \le \frac{n_*}{n_*-2}$, where the dimension like numbers are defined as

$$n^* = \frac{\Lambda_1}{\Lambda_2}(n-1) + 1,$$

$$n_* = \frac{\Lambda_2}{\Lambda_1}(n-1) + 1.$$

In order to understand the solution structure for (1.11), Felmer and Quass [**13**] consider (1.11) in the case of radially symmetric solutions. Using phase plane analysis, they establish that

Theorem A (i): For (1.11) with the Pucci extremal operator $\mathcal{M}^+_{\Lambda_1,\Lambda_2}$, there exists no non-trivial radial solution if $1 < p < p^*_+$ and $n^* > 2$, where

$$\max\{\frac{n^*}{n^*-2}, \frac{n+2}{n-2}\} < p^*_+ < \frac{n^*+2}{n^*-2}.$$

(ii): For (1.11) with the Pucci extremal operator $\mathcal{M}^-_{\Lambda_1,\Lambda_2}$, there exists no non-trivial radial solution if $1 < p < p^*_-$, where

$$\frac{n_*+2}{n_*-2} < p^*_- < \frac{n+2}{n-2}.$$

An explicit expression for p^*_+, p^*_- in term of Λ_1, Λ_2, n are still unknown. In order to obtain the full range of the exponent p for the Liouville-type theorem in (1.11), it is interesting to prove that the solutions in (1.11) are radially symmetric.

We first consider the radial symmetry for the fully nonlinear equations with general operator $F(D^2u)$ and show that

Theorem 1. *Assume $F(D^2u)$ satisfies (1.8) and (1.9). Let $n^* > 2$. If $u \in C(\mathbb{R}^n)$ be a nonnegative non-trivial solution of (1.1) and*

$$u = o(|x|^{-\frac{2}{p-1}}) \quad as\ |x| \to \infty \tag{1.13}$$

for $p > 1$, then u is radially symmetric and strictly decreasing about some point.

In the same spirit of the proof in Theorem 1, our conclusions also hold for general function $f(u)$, i.e.

$$F(D^2u) + f(u) = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.14}$$

Corollary 1. *Assume that $F(D^2u)$ satisfies (1.8) and (1.9), and*

(1.15)
$$\frac{f(u)-f(v)}{u-v} \le c(|u|+|v|)^\alpha, \text{ for } u,v \text{ sufficiently small, and some } \alpha > \frac{2}{n^*-2} \text{ and } c > 0.$$

Let $n^* > 2$ and u be a positive solution of (1.14) with

$$u(x) = O(|x|^{2-n^*}) \tag{1.16}$$

at infinity. Then u is radially symmetric and strictly decreasing about some point in $\mathbb{R}^n$.

Once the radial symmetry property of solutions is established, with the help of Theorem A, we immediately have the following corollary. We hope that our symmetry results shed some light on the complicated problem of Liouville-type theorems in (1.11) for the full range of the exponent p.

Corollary 2. *(i) For (1.11) with the Pucci extremal operator $\mathcal{M}^+_{\Lambda_1,\Lambda_2}$, there exists no non-trivial nonnegative solution satisfying (1.13) if $1 < p < p^*_+$ and $n^* > 2$.*
(ii) For (1.11) with the Pucci extremal operator $\mathcal{M}^-_{\Lambda_1,\Lambda_2}$, there exists no non-trivial nonnegative solution satisfying (1.13) if $1 < p < p^*_-$ and $n^* > 2$.

In carrying out the moving plane method, the maximum principle plays a crucial role. In order to adapt the moving plane method to non-proper fully nonlinear equations (that is, $F(x, u, Du, D^2u)$ is not nondecreasing in u), a new maximum principle has to be established for viscosity solutions.

In this paper, we will establish a new maximum principle for viscosity solutions to the equation

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2u) - \gamma|Du| + c(x)u \le 0 \quad \text{in } \Omega,$$

where $c(x) \in L^\infty(\Omega)$ is not necessarily negative. Similar maximum principle for classical solutions to semilinear equations was given in [**3**]. Since we consider the viscosity solutions here instead of classical solutions in [**3**], considerably more difficulties have to be taken care of in our case. Unlike the pointwise argument in [**3**], we apply the Hopf lemma for viscosity solutions in those minimum points. More specifically, we have

THEOREM 2. *Let Ω be a bounded domain. Assume that $\lambda(x), c(x) \in L^\infty(\Omega), \gamma \ge 0$, and $\psi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a positive solution in $\bar{\Omega}$ satisfying*

$$\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) + \lambda(x)\psi \le 0. \tag{1.17}$$

Let u be a viscosity solution of

$$\begin{cases} \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2u) - \gamma|Du| + c(x)u \le 0 & \text{in } \Omega, \\ u \ge 0 & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

If

$$c(x) \le \lambda(x) - \gamma|D\psi|/\psi, \tag{1.19}$$

then $u \ge 0$ in Ω.

Note that the function $c(x)$ may not be needed to be negative in order that the maximum principle holds. We also would like to point out the ψ is a supersolution of the equation involving $\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi)$ instead of $\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\psi)$. If a specific $\psi(x)$ is chosen, we can get the explicit control for $c(x)$ in order to obtain the maximum principle for (1.18). We are also able to extend the maximum principle to unbounded domains. We refer to Section 3 for more details.

Recently Caffarelli, Li and Nirenberg [**9**] [**10**] investigated the following problem

$$\begin{cases} \triangle u + f(u) = 0 & \text{in } \mathbb{B}\backslash\{0\}, \\ u = 0 & \text{in } \partial\mathbb{B} \end{cases} \tag{1.20}$$

in the case that f is locally Lipschitz. They obtained the radial symmetry and monotonicity property of solutions using an idea of Terracini [**22**]. Their results are also extended to fully nonlinear equations $F(x, u, Du, D^2u) = 0$ with differentiable components for $u \in C^2(\bar{\mathbb{B}}\backslash\{0\})$. However, this prevents us from applying these results to important classes of equations such as equations involving Pucci's extremal operators, Hamilton-Jacobi-Bellman or Isaacs equations. A maximum principle in a punctured domain is established in [**9**] in order to apply the moving plane technique. However, their maximum principle only holds in sufficiently small domains, since sufficient smallness of the domain is used in the spirit of Alexandroff-Pucci-Belman maximum principle (see [**2**]).

In this paper, we consider

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2u) - \gamma|Du| + c(x)u \le 0 \quad \text{in } \Omega\backslash\{0\}.$$

We will obtain a new maximum principle in terms of the assumption of $c(x)$ (see Lemma 10). It is especially true for a sufficiently small domain just as Caffarelli, Li and Nirenberg's maximum principle, since the bound of $c(x)$ in Lemma 10 preserves automatically if $|x|$ is small enough. Our result is not only an extension for viscosity solutions, but also an interesting result for semilinear elliptic equations. Furthermore, we obtain the radial symmetry of solutions in a punctured ball.

Theorem 3. *Let $u \in C(\mathbb{B}\backslash\{0\})$ be a positive viscosity solution of (1.2) in the case that $f(u)$ is locally Lipschitz. Then u is radially symmetric with respect to the origin and u is strictly decreasing in $|x|$.*

Finally, we consider the radial symmetry of the Cauchy problem for viscosity solutions of the fully nonlinear parabolic equation (1.3). C. Li [**16**] obtain the monotonicity and radial symmetry properties of classic solution $u \in C^2(\mathbb{R}^n \times (0,T])$ for fully nonlinear parabolic equations $\partial_t u - F(x,u,Du,D^2u) = 0$ with differentiable components. Again this result does not apply to fully nonlinear parabolic equations involving Pucci extremal operators, Hamilton-Jacobi-Bellman or Iassac equations. For further extensions about asymptotic symmetry or radial symmetry of entire solutions, etc. for parabolic problems on bounded or unbounded domains, we refer to the survey of Polá*č*ik [**19**].

In this paper, we prove that

Theorem 4. *Let $u \in C(\mathbb{R}^n \times (0,T])$ be a positive viscosity solution of (1.3). Assume that*

$$|u(x,t)| \to 0 \quad \textit{uniformly as } |x| \to \infty, \tag{1.21}$$

and

$$u_0(x_1,x') \le u_0(y,x') \textit{ for } x_1 \le y \le -x_1,\ x_1 \le 0 \textit{ and } x' = (x_2,\cdots,x_n).$$

Then u is nondecreasing in x_1 and $u(x_1,x',t) \le u(-x_1,x',t)$ for $x_1 \le 0$. Furthermore, if $u_0(x)$ is radially symmetric with respect to the origin and nonincreasing in $|x|$, then $u(x)$ is radial symmetry with respect to $(0,t)$ for each fixed $t \in (0,T]$ and nonincreasing in $|x|$.

The outline of the paper is as follows. In Section 2, we present the basic results for the definition of viscosity solutions, the strong maximum principle and the maximum principle in a small domain for viscosity solutions, etc. Section 3 is devoted to providing the proof of Theorem 1 and Theorem 2. New maximum principles and their extensions are established. The radial symmetry of solutions in a punctured ball and the corresponding maximum principle are obtained in Section 4. In Section 5, we prove the radial symmetry for viscosity solutions of fully nonlinear parabolic equations. Throughout the paper, The letters C, c denote generic positive constants, which is independent of u and may vary from line to line.

2. Preliminaries

In this section we collect some basic results which will be applied through the paper for fully nonlinear partial differential equations. We refer to [**6**], [**7**], [**8**], and references therein for a detailed account.

Let us recall the notion of viscosity sub and supersolutions of the fully nonlinear elliptic equation

$$F(Du, D^2u) + f(u) = 0 \quad \text{ in } \Omega, \tag{2.1}$$

where Ω is an open domain in $\mathbb{R}^n$ and $F : \mathbb{R}^n \times S_n(\mathbb{R}) \to \mathbb{R}$ is a continuous map with $F(p,M)$ satisfying *(F1)*.

Definition: A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution (subsolution) of (2.1) in Ω, when the following condition holds: If $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ and $u - \phi$ has a local minimum (maximum) at x_0, then

$$F(D\phi(x_0), D^2\phi(x_0)) + f(u(x_0)) \leq (\geq) 0.$$

If u is a viscosity supersolution (subsolution), we say that u verifies

$$F(Du, D^2u) + f(u) \leq (\geq) 0$$

in the viscosity sense. We say that u is a viscosity solution of (2.1) when it simultaneously is a viscosity subsolution and supersolution.

We also present the notion of viscosity sub and supersolutions of the fully nonlinear parabolic equation (see e.g. [**23**])

$$\partial_t u - F(Du, D^2u) - f(u) = 0 \quad \text{in } \Omega_T := \Omega \times (0, T]. \tag{2.2}$$

Definition: A continuous function $u : \Omega_T \to \mathbb{R}$ is a viscosity supersolution (subsolution) of (2.2) in Ω_T, when the following condition holds: If $(x_0, t_0) \in \Omega_T$, $\phi \in C^2(\Omega_T)$ and $u - \phi$ has a local minimum (maximum) at (x_0, t_0), then

$$\partial_t \phi(x_0, t_0) - F(D\phi(x_0, t_0), D^2\phi(x_0, t_0)) - f(u(x_0, t_0)) \geq (\leq) 0.$$

We say that u is viscosity solution of (2.2) when it both is a viscosity subsolution and supersolution.

We state a strong maximum principle and the Hopf lemma for non-proper operators in fully nonlinear elliptic equations (see e.g. [**1**]).

LEMMA 1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and let $b(x), c(x) \in L^\infty(\Omega)$. Suppose $u \in C(\bar{\Omega})$ is a viscosity solution of*

$$\begin{cases} \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2u) - b(x)|Du| + c(x)u \leq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Then either $u \equiv 0$ in Ω or $u > 0$ in Ω. Moreover, at any point $x_0 \in \partial\Omega$ where $u(x_0) = 0$, we have

$$\liminf_{t \to 0} \frac{u(x_0 + t\nu) - u(x_0)}{t} < 0,$$

where $\nu \in \mathbb{R}^n \backslash \{0\}$ is such that $\nu \cdot n(x_0) > 0$ and $n(x_0)$ denotes the exterior normal to $\partial\Omega$ at x_0.

It is straightforward to deduce the strong maximum principle for proper operators in fully nonlinear elliptic equations from the Hopf Lemma.

LEMMA 2. *Let $\Omega \in \mathbb{R}^n$ be an open set and let $u \in C(\Omega)$ be a viscosity solution of*

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2u) - b(x)|Du| + c(x)u \leq 0$$

with $b(x), c(x) \in L^\infty(\Omega)$ and $c(x) \leq 0$. Suppose that u achieves a non-positive minimum in Ω. Then u is a constant.

We shall make use of the following maximum principle which does not depend on the sign of $c(x)$, but instead, on the measure of the domain Ω (see e.g. [**12**]).

LEMMA 3. *Consider a bounded domain Ω and assume that $|c(x)| < m$ in Ω and $\gamma \geq 0$. Let $u \in C(\bar{\Omega})$ be a viscosity solution of*

$$\begin{cases} \mathcal{M}^{-}_{\Lambda_1,\Lambda_2}(D^2u) - \gamma|Du| + c(x)u \leq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists a constant $\delta = \delta(\Lambda_1, \Lambda_2, \gamma, n, m, diam(\Omega))$ such that we have $u \geq 0$ in Ω provided $|\Omega| < \delta$.

The following result is concerned about the regularity of viscosity solutions in [**6**].

LEMMA 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $(F1)$ is satisfied and f is locally Lipschitz. Let $u \in C(\bar{\Omega})$ be a viscosity solution of*

$$F(Du, D^2u) + f(u) = 0 \quad \text{in } \Omega.$$

Then u is in $C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$.

In the process of employing the moving plane method, we need to compare u at x with its value at its reflection point of x. The next lemma shows that the difference of a supersolution and a subsolution of the fully nonlinear equation is still a supersolution. Unlike the case of the classical solutions of fully nonlinear equations $F(x, u, Du, D^2u) = 0$ with differentiable components, the difficulty here is the lack of regularity of u. The following result is first showed in [**12**]. We also refer the reader to [**11**] and [**18**] for related results. In Section 4, we will derive similar results for viscosity solutions of fully nonlinear parabolic equations.

LEMMA 5. *Assume that $F(Du, D^2u)$ satisfies $(F1)$ and f is locally Lipschitz. Let $u_1 \in C(\bar{\Omega})$ and $u_2 \in C(\bar{\Omega})$ be respectively a viscosity subsolution and supersolution of*

$$F(Du, D^2u) + f(u) = 0 \qquad \text{in } \Omega.$$

Then the function $v = u_2 - u_1$ is a viscosity solution of

$$\mathcal{M}^{-}_{\Lambda_1,\Lambda_2}(D^2v) - \gamma|Dv| + c(x)v(x) \leq 0,$$

where

$$c(x) = \begin{cases} \frac{f(u_2(x)) - f(u_1(x))}{u_2(x) - u_1(x)}, & \text{if } u_2(x) \neq u_1(x), \\ \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

In the proof of Lemma 5 in [**12**], an equivalent definition of viscosity solutions in terms of semijets is used (see [**8**]). In order to obtain the parabolic version of Lemma 5, we denote by $\mathcal{P}^{2,+}_{\Omega}, \mathcal{P}^{2,-}_{\Omega}$ the parabolic semijets.

Definition:

$$\begin{aligned} \mathcal{P}^{2,+}_{\Omega}u(z,s) = \; & \{(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S_n(\mathbb{R}) : u(x,t) \leq u(z,s) + a(t-s) + <p, x-z> + \\ & \tfrac{1}{2} <X(x-z), x-z> + o(|t-s| + |x-z|^2) \text{ as } \Omega_T \ni (x,t) \to (z,s)\}. \end{aligned} \tag{2.4}$$

While we define $P^{2,-}_{\Omega}(u) := -P^{2,+}_{\Omega}(-u)$.

3. Symmetry of Viscosity Solutions in $\mathbb{R}^n$

In this section, we will obtain the radial symmetry of nonnegative solution in (1.1). We first present a technical lemma about the eigenvalue of a radial function. It could be verified by a direct calculation.

Lemma 6. *Let $\psi : (0,\ +\infty) \to \mathbb{R}$ be a C^2 radial function. For $\forall\ x \in \mathbb{R}^n \backslash \{0\}$, the eigenvalues of $D^2\psi(|x|)$ are $\psi''(|x|)$, which is simple and $\frac{\psi'(|x|)}{|x|}$, which has multiplicity $(n-1)$.*

Based on the above conclusion, we may select specific functions. For instance, let $\psi = |x|^{-q}$ and $0 < q < n^* - 2$. Recall that $n^* = \frac{\Lambda_1}{\Lambda_2}(n-1)+1$. The eigenvalues are $q(q+1)|x|^{-q-2}$ and $-q|x|^{-q-2}$. From the above lemma, for $x \in \mathbb{R}^n \backslash \{0\}$,

$$\begin{aligned} \mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi)(x) &= \Lambda_2 q(q+1)|x|^{-q-2} - \Lambda_1(n-1)q|x|^{-q-2} \\ &= \tfrac{q(\Lambda_2(q+1)-\Lambda_1(n-1))}{|x|^2}\psi(x). \end{aligned}$$

Notice that $0 < q < n^* - 2$ implies that

$$q(\Lambda_2(q+1) - \Lambda_1(n-1)) < 0.$$

We shall make use of a simple lemma, which enables us to consider the product of a viscosity solution and an auxiliary function. The argument is in the spirit of Lemma 2.1 in [**12**]. However, the idea behind it is different. In their lemma, $u(x)$ is assumed to be nonnegative. We do not impose this assumption. In other words, we specifically focus on the points where $u(x)$ is negative.

Lemma 7. *Let $u \in C(\Omega)$ satisfy*

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2u) - b(x)|Du| + c(x)u \le 0 \tag{3.1}$$

where $b(x), c(x) \in L^\infty(\Omega)$. Suppose $\psi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is strictly positive in $\bar{\Omega}$. Assume $u(x_0) < 0$. Then $\bar{u} := u/\psi$ satisfies

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\bar{u}) - \bar{b}(x)|D\bar{u}| + \bar{c}(x)\bar{u} \le 0 \tag{3.2}$$

at x_0, where

$$\bar{b}(x) = \frac{2\sqrt{n}\Lambda_2|D\psi|}{\psi} + |b|$$

and

$$\bar{c}(x) = c(x) + \frac{\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) + |b||D\psi|}{\psi}.$$

Proof. Let $\phi(x) \in C^2(\Omega)$ be the test function that toughes $\bar{u}$ from below at x_0, that is $\phi(x_0) = \bar{u}(x_0)$ and $\bar{u}(x) \ge \phi(x)$ in Ω. Then $u(x_0) = \phi(x_0)\psi(x_0)$ and $u(x) \ge \phi(x)\psi(x)$ in Ω, which indicates that $\phi(x)\psi(x)$ toughes u from below. Simple calculations show that

$$D(\phi\psi) = D\phi \cdot \psi + D\psi \cdot \phi,$$

$$D^2(\phi\psi) = \phi D^2\psi + 2D\phi \otimes D\psi + D^2\phi\psi,$$

where $\otimes$ denotes the symmetric tensor product with $p \otimes q = \frac{1}{2}(p_iq_j + p_jq_i)_{i,j}$. By the properties of the Pucci extremal operators, we have

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(M+N) \ge \mathcal{M}^-_{\Lambda_1,\Lambda_2}(M) + \mathcal{M}^-_{\Lambda_1,\Lambda_2}(N),$$

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(aM) = a\mathcal{M}^+_{\Lambda_1,\Lambda_2}(M)$$

for $a \le 0$. We also note that

$$tr(A(p\otimes q)) \le |A||p\otimes q| \le \sqrt{n}\Lambda_2|p||q|,$$

where A is a matrix whose eigenvalues lie in $[\Lambda_1, \Lambda_2]$ and $|A| := \sqrt{tr(A^TA)}$. Since $\phi\psi$ is a test function for u and $\phi(x_0) = \bar{u}(x_0) < 0$, taking into account the above properties, we get

$$\begin{aligned} 0 &\ge c(x)\phi\psi - b|D(\phi\psi)| + \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2(\psi\phi)) \\ &\ge c(x)\phi\psi + |b||D\psi|\phi - |b||D\phi|\psi + \psi\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\phi) - 2\sqrt{n}\Lambda_2|D\phi||D\psi| \\ &\quad +\phi\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) \\ &\ge (c(x)\psi + \mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) + |b||D\psi|)\phi - (2\sqrt{n}\Lambda_2|D\psi| + |b|\psi)|D\phi| + \psi\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\phi) \end{aligned}$$

at x_0. Dividing both sides by ψ, we obtain

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\phi)(x_0) - \bar{b}(x_0)|D\phi|(x_0) + \bar{c}(x_0)\phi(x_0) \le 0,$$

where $\bar{b}(x), \bar{c}(x)$ are in the statement of the lemma.

□

Using the above lemma and the strong maximum principle in Lemma 2, we are able to consider the maximum principle in terms of $c(x)$ for non-proper operators in fully nonlinear elliptic equation.

PROOF OF THEOREM 2. We prove it by contradiction argument. Suppose that $u(x) < 0$ somewhere in Ω. Let

$$\bar{u}(x) = \frac{u(x)}{\psi(x)}.$$

Then $\bar{u}(x) < 0$ somewhere in Ω. Since $u(x) \ge 0$ on $\partial\Omega$, we may assume that $\bar{u}(x^*) = \inf_\Omega \bar{u}(x) < 0$, where $x^* \in \Omega$. By the continuity of $\bar{u}(x)$, we can find a connected neighborhood Ω' containing x^* such that $\bar{u}(x) < 0$ in Ω' and $\bar{u}(x) \not\equiv u(x^*)$ in Ω'. Otherwise, $u(x) \equiv u(x^*)$ in Ω, it is obviously a contradiction. Thanks to Lemma 7 with $b(x)$ replaced by γ, $\bar{u}$ satisfies

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\bar{u}) - \bar{b}(x)|D\bar{u}| + \bar{c}(x)\bar{u} \le 0 \quad \text{in } \Omega'. \tag{3.3}$$

Recall that

$$\bar{b}(x) = \frac{2\sqrt{n}\Lambda_2|D\psi|}{\psi} + \gamma \in L^\infty(\Omega')$$

and

$$\bar{c}(x) = c(x) + \frac{\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) + \gamma|D\psi|}{\psi} \in L^\infty(\Omega').$$

By the assumptions (1.19) and (1.17),

$$c(x) + \frac{\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) + \gamma|D\psi|}{\psi} \le 0.$$

Thanks to the strong maximum principle in Lemma 2, $\bar{u}(x) \equiv \bar{u}(x^*)$ in Ω'. It contradicts our assumption. This contradiction leads to the proof of the lemma. □

REMARK 1. *1. From the proof, we can see that the same reasoning follows when the condition (1.17) and (1.19) hold where u is negative.*

2. If $c(x), \lambda(x)$ are continuous, we only need $c(x^*) < \lambda(x^*) - \gamma|D\psi|/\psi(x^*)$, where x^* is the point where u reaches minimum.

In the spirit of the above argument, we extend the corresponding maximum principle to unbounded domains. We need to guarantee that the minimum is only achieved in the interior of the domain.

LEMMA 8. *Let Ω be an unbounded domain. If u, ψ satisfy the same conditions as that in Theorem 2 and assume that*

$$\liminf_{|x|\to\infty} \frac{u(x)}{\psi(x)} \geq 0, \tag{3.4}$$

then $u \geq 0$ in Ω.

PROOF. Note that the assumption (3.4) implies that the minimum of u/ψ will not go to infinity. Then the minimum of u/ψ lies only in the interior of Ω. Applying the same argument as in the proof of Theorem 2, the conclusion follows. □

If some particular $\psi(x)$ is given, then $c(x)$ could be controlled explicitly, which is especially useful in applying the maximum principle. We call the following useful maximum principle as "Decay at infinity".

COROLLARY 3. *(Decay at infinity) Assume that there exists $R > 0$ such that*

$$c(x) \leq \frac{-q(\Lambda_2(q+1) - \Lambda_1(n-1))}{|x|^2} - \frac{\gamma q}{|x|} \qquad \text{for } |x| > R \tag{3.5}$$

and

$$\liminf_{|x|\to\infty} u(x)|x|^q \geq 0. \tag{3.6}$$

Let Ω be a region in $\mathbb{B}_R^c(0) = \mathbb{R}^n \backslash \mathbb{B}_R(0)$. If u satisfies (1.18) in Ω, then

$$u(x) \geq 0 \qquad \text{for all } \ x \in \Omega.$$

PROOF. We consider the specific function $\psi(x) = |x|^{-q}$, As we know,

$$\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi)(x) - \frac{q(\Lambda_2(q+1) - \Lambda_1(n-1))}{|x|^2}\psi(x) = 0.$$

Applying Lemma 8, we conclude the proof. □

REMARK 2. *i)It is similar to Remark 1, the conclusion holds when (3.5) is true at points where u is negative.*

ii) In the case of $\gamma = 0$, $c(x) \leq \frac{-q(\Lambda_2(q+1)-\Lambda_1(n-1))}{|x|^2}$. Notice that $c(x)$ may not be needed to be negative in order that the maximum principle holds.

In the rest of this section, we are going to adapt the moving plane technique in the viscosity solution setting to prove Theorem 1. We refer to the book [**3**] for more account of the moving plane method in semilinear elliptic equations. Before we carry out the moving plane method, we introduce several necessary notations. Set

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_1 < \lambda\}$$

and $T_\lambda = \partial\Sigma_\lambda$. Define x^λ be the reflection of x with respect to T_λ, i.e. $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$. Let

$$u_\lambda(x) = u(x^\lambda)$$

and

$$v_\lambda(x) = u_\lambda(x) - u(x).$$

The moving plane method to obtain the radial symmetry consists of two steps. In the first step, we show that the plane can move, that is, we will deduce that, for sufficiently negative λ,

$$v_\lambda(x) \geq 0, \qquad \forall x \in \Sigma_\lambda, \tag{3.7}$$

where we are going to use the corollary of decay at infinity. In the second step, we will move the plane T_λ to the right as long as (3.7) holds. The plane will stop at some critical position, say at $\lambda = \lambda_0$. We will verify that

$$v_{\lambda_0} \equiv 0, \qquad \forall x \in \Sigma_{\lambda_0}. \tag{3.8}$$

These two steps imply that $u(x)$ is symmetric and monotone decreasing about the plane T_{λ_0}. Since the equation (1.1) is invariant under rotation, we can further infer that $u(x)$ must be radially symmetric with respect to some point.

Proof of Theorem 1: We derive the proof in two steps.

Step 1: By the hypothesis (1.9), u_λ satisfies the same equation as u does. Thanks to Lemma 5 for the case $\gamma = 0$,

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 v_\lambda) + p\psi_\lambda^{p-1}(x)v_\lambda(x) \leq 0, \tag{3.9}$$

where $\psi_\lambda(x)$ is between $u_\lambda(x)$ and $u(x)$. In order to apply the corollary of decay at infinity, by (ii) in Remark 2, it is sufficient to verify that

$$\psi_\lambda^{p-1}(x) \leq \frac{C}{|x|^2} \tag{3.10}$$

and

$$\liminf_{|x|\to\infty} v_\lambda(x)|x|^q \geq 0. \tag{3.11}$$

For (3.10), to be more precise, we only need to show that (3.10) holds at the points $\tilde{x}$ where v_λ is negative (see Remark 2). At those points,

$$u_\lambda(\tilde{x}) < u(\tilde{x}).$$

Then

$$0 \leq u_\lambda(\tilde{x}) \leq \psi_\lambda(\tilde{x}) \leq u(\tilde{x}).$$

By the decay assumption (1.13), we derive that

$$c(\tilde{x}) = p\psi_\lambda^{p-1}(\tilde{x}) \leq o(|\tilde{x}|^{-2}) \leq C|\tilde{x}|^{-2},$$

that is, (3.10) is satisfied. Note that the fact λ is sufficiently close to negative infinity is applied. By the decay assumption (1.13) again, for any small ϵ,

$$\liminf_{|x|\to\infty} v_\lambda(x)|x|^q \geq \liminf_{|x|\to\infty} -u(x)|x|^q \geq \liminf_{|x|\to\infty} \frac{-\epsilon}{|x|^{\frac{2}{p-1}-q}}.$$

If $\frac{2}{p-1} - q > 0$, then (3.11) is fulfilled. Hence we fixed $0 < q < \min\{\frac{2}{p-1}, (n^* - 2)\}$.

Step 2: We continue to move the plane T_λ to the right as long as (3.7) holds. Define

$$\lambda_0 = \sup\{\lambda \; | v_\mu(x) \geq 0 \text{ in } \Sigma_\mu \text{ for every } \; \mu \leq \lambda\}.$$

Since $u(x) \to 0$ as $|x| \to \infty$, we infer that $\lambda_0 < \infty$. If $\lambda_0 > 0$, by the translation invariance of the equation, we may do a translation to let the critical position be negative. If $\lambda_0 = 0$, we move the plane from the positive infinity to the left. If $\lambda_0 = 0$ again, we obtain the symmetry of the solution at $x_1 = 0$. In all the cases, we may consider Σ_{λ_0} with $\lambda_0 < 0$, which avoids the singularity of $\psi(x) = |x|^{-q}$ at the origin. Our goal is to show that $v_{\lambda_0}(x) \equiv 0$ in Σ_{λ_0}. Otherwise, by the strong maximum principle in Lemma 1, we have $v_{\lambda_0} > 0$ in Σ_{λ_0}. If this is the case, we will show that the plane can continue to move to the right a little bit more, that is, there exists a ϵ_0 such that, for all $0 < \epsilon < \epsilon_0$, we have

$$v_{\lambda_0+\epsilon} \geq 0, \quad \forall x \in \Sigma_{\lambda_0+\epsilon}. \tag{3.12}$$

It contradicts the definition of λ_0. Therefore, (3.8) must be true. Set

$$\bar{v}_\lambda(x) := \frac{v_\lambda(x)}{\psi(x)}.$$

Suppose that (3.12) does not hold, then there exist a sequence of ϵ_i such that $\epsilon_i \to 0$ and a sequence of $\{x^i\}$, where $\{x^i\}$ is the minimum point such that

$$\bar{v}_\lambda(x) = \liminf_{\Sigma_{\lambda_0+\epsilon_i}} v_\lambda(x).$$

We claim that there exists a $\bar{R}$ such that $|x^i| < \bar{R}$ for all i. For a clear presentation, this claim is verified in Lemma 9 below. By the boundedness of $\{x^i\}$, there exists a subsequence of $\{x^i\}$ converging to some point $x^0 \in \Sigma_{\lambda_0}$. Since

$$\bar{v}_{\lambda_0}(x^0) = \lim_{i\to\infty} \bar{v}_{\lambda_0+\epsilon_i}(x_i) \leq 0$$

and $\bar{v}_{\lambda_0}(x) > 0$ for $x \in \Sigma_{\lambda_0}$, we obtain that $x^0 \in T_{\lambda_0}$ and $\bar{v}_{\lambda_0}(x^0) = 0$. By the regularity of fully nonlinear equations in Lemma 4 and the fact that $\psi(x) \in C^2(\Sigma_{\lambda_0})$, we know that at least $\bar{v}_\lambda(x) \in C^1(\Sigma_{\lambda_0})$. Consequently,

$$\nabla \bar{v}_{\lambda_0}(x^0) = \lim_{i\to\infty} \nabla \bar{v}_{\lambda_0+\epsilon_i}(x^i) = 0.$$

It follows that

$$\nabla v_{\lambda_0}(x^0) = \nabla \bar{v}_{\lambda_0}(x^0)\psi(x^0) + \bar{v}_{\lambda_0}(x^0)\nabla\psi(x^0) = 0. \tag{3.13}$$

Since $v_{\lambda_0}(x^0) = 0$ and $v_{\lambda_0}(x) > 0$ for $x \in \Sigma_{\lambda_0}$, thanks to the Hopf lemma (i.e. Lemma 1) , we readily get that

$$\frac{\partial v_{\lambda_0}}{\partial n}(x^0) < 0,$$

where n is the outward normal at T_{λ_0}. It is a contradiction to (3.13). In the end, we conclude that $u_{\lambda_0}(x) \equiv u(x)$, i.e. (3.8) holds.

□

The following lemma verifies the claim in the proof of Theorem 1.

Lemma 9. *There exists a $\bar{R}$ (independent of λ) such that $|x_0| < \bar{R}$, where x_0 is the point where $\bar{v}_\lambda(x)$ achieves the minimum and $\bar{v}_\lambda(x_0) < 0$.*

PROOF. If $|x_0|$ is sufficiently large, by the decay rate of u,

$$c(x_0) = p\psi_\lambda^{p-1}(x_0) < C|x_0|^{-2} = -\frac{\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi)(x_0)}{\psi(x_0)}, \tag{3.14}$$

where $C = -q(\Lambda_2(q+1) - \Lambda_1(n-1)) > 0$ and $\psi(x) = |x|^{-q}$. It follows from the argument of Theorem 2 in the case of $\gamma = 0$ that

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\bar{v}_\lambda) - \bar{b}(x)|D\bar{v}_\lambda| + \bar{c}(x)\bar{v}_\lambda \leq 0 \quad \text{in } \Sigma_\lambda.$$

Here

$$\bar{b}(x) = \frac{2\sqrt{n}\Lambda_2|D\psi|}{\psi}$$

and

$$\bar{c}(x) = c(x) + \frac{\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi)}{\psi}.$$

From (3.14), we see that there exists a neighborhood Ω' of x_0 such that $\bar{c}(x) < 0$ in Ω'. The strong maximum principle in Lemma 2 further implies that

$$\bar{v}_\lambda(x) \equiv \bar{v}_\lambda(x_0) < 0 \quad \text{for } |x| > |x_0|. \tag{3.15}$$

On the other hand,

$$\bar{v}_\lambda(x) = [o(|x_\lambda|^{-\frac{2}{p-1}}) - o(|x|^{-\frac{2}{p-1}})|x|^q] \to 0$$

as $|x| \to \infty$, which contradicts (3.15). Hence the lemma is completed. □

PROOF OF COROLLARY 1: Adopting the same notations in the proof of Theorem 1, for the general function $f(u)$, we have

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 v_\lambda) + c_\lambda(x)v_\lambda(x) \leq 0, \tag{3.16}$$

where

$$c_\lambda(x) = \begin{cases} \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)}, & \text{if } u_\lambda(x) \neq u(x), \\ 0, & \text{otherwise.} \end{cases} \tag{3.17}$$

As argued in Theorem 1, we should verify that

$$c_\lambda(x) \leq \frac{C}{|x|^2} \tag{3.18}$$

and

$$\liminf_{|x|\to\infty} v_\lambda(x)|x|^q \geq 0. \tag{3.19}$$

We only need to focus on the points $\tilde{x}$ where $u_\lambda(\tilde{x}) < u(\tilde{x})$ for (3.18). From the assumption (1.15), if $\tilde{x}$ is large enough,

$$c_\lambda(\tilde{x}) \leq c(|u_\lambda| + |u|)^\alpha(\tilde{x}) = O(|\tilde{x}|^{(2-n^*)\alpha}) \leq \frac{C}{|\tilde{x}|^2} \tag{3.20}$$

for $\alpha > \frac{2}{n^*-2}$. Recall that $n^* = \frac{\Lambda_1}{\Lambda_2}(n-1) + 1$. Since $u(x)$ is positive, then

$$v_\lambda(x)|x|^q > -u(x)|x|^q.$$

If $u(x) = O(|x|^{2-n^*})$, then

$$\liminf_{|x|\to\infty} v_\lambda(x)|x|^q \geq \liminf_{|x|\to\infty} -u(x)|x|^q = 0$$

for $0 < q < n^* - 2$. Hence (3.18) and (3.19) are satisfied. The rest of proof follows from the same argument in Theorem 1. $\square$

4. Symmetry of Viscosity Solutions in a Punctured Ball

In this section, we consider the radial symmetry of viscosity solutions in a punctured ball. Due to the singularity of the point, the corresponding maximum principle shall be established. Instead of only considering sufficiently small domains, our result is valid under the appropriate upper bound of $c(x)$. The result also holds if $c(x)$ is bounded and the domain is appropriately small. Thanks to Lemma 5, we only consider the following equation.

$$\mathcal{M}^{-}_{\Lambda_1,\Lambda_2}(D^2u) - \gamma|Du| + c(x)u \le 0 \quad \text{in } \Omega\backslash\{0\}. \tag{4.1}$$

Lemma 10. *Let Ω be a connected and bounded domain in $\mathbb{R}^n$ and u be the viscosity solution of (4.1). Assume that $c(x) \in L^\infty(\Omega\backslash\{0\})$, and*

(4.2)
$$\begin{cases} c(x) \le \frac{q(\Lambda_1(n-1)-\Lambda_2(q+1))}{|x|^2} - \frac{\gamma q}{|x|} \text{ with } 0 < q < n^* - 2 & \text{if } n^* > 2, \\ \text{or} & \\ c(x) \le \Lambda_2/4(-\ln|x|)^{-2}|x|^{-2} - \gamma/2(-\ln|x|)^{-1}|x|^{-1} \text{ with } |x| \le 1 \text{ in } \Omega & \text{if } n^* = 2. \end{cases}$$

Moreover, u is bounded from below and $u \ge 0$ on $\partial\Omega$. Then $u \ge 0$ in $\Omega\backslash\{0\}$.

Proof. Our proof is based on the idea in Theorem 2. Recall again that $n^* = \frac{\Lambda_1}{\Lambda_2}(n-1)+1$. If $n^* > 2$, let $\psi(x) = |x|^{-q}$. If $n^* = 2$, we select $\psi(x) = (-\ln|x|)^a$, where $0 < a < 1$. Set

$$\bar{u}(x) := \frac{u(x)}{\psi(x)}.$$

Since u is bounded from below in $\Omega\backslash\{0\}$ and $\psi(x) \to \infty$ as $|x| \to 0$, then

$$\liminf_{|x|\to 0} \bar{u}(x) \ge 0.$$

It is easy to know that $\bar{u}(x) \ge 0$ on $\partial\Omega$. Suppose $u(x) < 0$ somewhere in $\Omega\backslash\{0\}$, then $\bar{u}(x) < 0$ somewhere in $\Omega\backslash\{0\}$. Hence $\inf_{\Omega\backslash\{0\}} \bar{u}(x)$ is achieved at some point $x_0 \in \Omega\backslash\{0\}$. Therefore, we can find a neighborhood Ω' of x_0 such that $\bar{u}(x) < 0$ and $\bar{u}(x) \not\equiv \bar{u}(x_0)$ in Ω'. Otherwise, $\bar{u}(x) \equiv \bar{u}(x_0)$ in Ω', which is obviously impossible. Recall in Theorem 2 that,

$$\mathcal{M}^{-}_{\Lambda_1,\Lambda_2}(D^2\bar{u}) - \bar{b}(x)|D\bar{u}| + \bar{c}(x)\bar{u} \le 0 \quad \text{in } \Omega', \tag{4.3}$$

where

$$\bar{c}(x) = c(x) + \frac{\mathcal{M}^{+}_{\Lambda_1,\Lambda_2}(D^2\psi) + \gamma|D\psi|}{\psi}.$$

In order to apply the strong maximum principle, we need $\bar{c}(x) \le 0$, i.e.

$$c(x) \le -\frac{\mathcal{M}^{+}_{\Lambda_1,\Lambda_2}(D^2\psi) + \gamma|D\psi|}{\psi}. \tag{4.4}$$

If $n^* > 2$, then $\psi(x) = |x|^{-q}$,

$$\frac{\mathcal{M}^{+}_{\Lambda_1,\Lambda_2}(D^2\psi) + \gamma|D\psi|}{\psi} = \frac{q(\Lambda_2(q+1) - \Lambda_1(n-1))}{|x|^2} + \frac{\gamma q}{|x|}.$$

Let

$$c(x) \leq \frac{q(\Lambda_1(n-1)-\Lambda_2(q+1))}{|x|^2} - \frac{\gamma q}{|x|}.$$

Then (4.4) is satisfied.

If $n^* = 2$, then $\psi(x) = (-\ln|x|)^a$,

$$\frac{\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\psi) + \gamma|D\psi|}{\psi} = \Lambda_2(a-1)a(-\ln|x|)^{-2}|x|^{-2} + \gamma a(-\ln|x|)^{-1}|x|^{-1}.$$

Hence we may assume that

$$c(x) \leq \Lambda_2/4(-\ln|x|)^{-2}|x|^{-2} - \gamma/2(-\ln|x|)^{-1}|x|^{-1}, \tag{4.5}$$

which implies that (4.4) holds for $a = 1/2$. If $c(x)$ is in the above range, by the strong maximum principle in Lemma 2, we readily deduce that $u(x) \equiv u(x_0)$ in Ω'. We then arrive at a contradiction. The proof of the lemma follows. $\square$

REMARK 3. *The assumption (4.2) is clearly satisfied when $|c(x)|$ is bounded and Ω is sufficiently small.*

With the above maximum principle in hand, we are able to prove the radial symmetry of viscosity solutions. We adapt the argument of [**10**] in our setting. Let the domain O be bounded and convex in direction of x_1, symmetric with respect to the hyperplane $\{x_1 = 0\}$. We prove the radial symmetry and monotonicity properties in O. Theorem 3 is an immediate consequence of Theorem 5 below. Let us first introduce several notations. Set

$$\Sigma_\lambda := \{x = (x_1, \cdots, x_n) \in O | x_1 < \lambda\}$$

and $T_\lambda = \{x \in O | x_1 = \lambda\}$. Define x^λ be the reflection of x with respect to T_λ. Let

$$u_\lambda(x) = u(x^\lambda)$$

and

$$v_\lambda(x) = u_\lambda(x) - u(x).$$

THEOREM 5. *Let $u \in C(\bar{O}\backslash\{0\})$ be a positive viscosity solution of*

$$F(Du, D^2u) + f(u) = 0 \qquad \text{in } \bar{O}\backslash\{0\}. \tag{4.6}$$

Then u is symmetric in x_1, that is, $u(x_1, x_2, \cdots, x_n) = u(-x_1, x_2, \cdots, x_n)$ for all $x \in O\backslash\{0\}$. In addition, u is strictly increasing in $x_1 < 0$.

PROOF OF THEOREM 5. Without loss of generality, we may assume that $\inf_O x_1 = -1$. We carry out the moving plane method in two steps.

Step 1: We show that the plane can move, i.e. there exists $-1 < \lambda_0 < -\frac{1}{2}$ such that

$$v_\lambda \geq 0 \ \text{ in } \Sigma_\lambda$$

for $-1 < \lambda < \lambda_0$. By (F2), u_λ satisfies the same equation as u does. Thanks to Lemma 5, we know that v_λ satisfies

$$\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2v_\lambda) - \gamma|Dv_\lambda| + c_\lambda(x)v_\lambda \leq 0 \quad \text{ in } O\backslash\{0\},$$

where

$$c_\lambda(x) = \begin{cases} \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)}, & \text{if } u_\lambda(x) \neq u(x), \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Since f is locally Lipschitz in $(0,\infty)$, then $|c_\lambda(x)| < C$ in O for some $C > 0$. It is clear that $v_\lambda(x) \geq 0$ in $\partial\Sigma_\lambda$. If λ is sufficient close to -1, then Σ_λ is small enough. By the maximum principle for small domains in Lemma 3, we readily deduce that $v_\lambda \geq 0$ in Σ_λ. Step 1 is then completed.

Define

$$\lambda_0 = \sup\{\lambda | -1 < \mu < 0,\ v_\mu \geq 0 \text{ in } \Sigma_\mu \backslash \{0^\mu\} \text{ for } \mu \leq \lambda < 0\}.$$

Step 2: We are going to show that $\lambda_0 = 0$. If it is true, we move the plane from the position where $\sup_O x_1 = 1$ to the left. By the symmetry of O, the plane will reach $\lambda_0 = 0$ again. Hence the symmetry of viscosity solutions is obtained. We divide the proof into three cases and show that the following cases are impossible to occur.

Case 1: $-1 < \lambda_0 < -\frac{1}{2}$.

If this is the case, we are going to show that the plane can still be moved a little bit more to the right. By the strong maximum principle in Lemma 1, we have $v_{\lambda_0}(x) > 0$ in Σ_{λ_0}. Set $\lambda = \lambda_0 + \epsilon$ for sufficiently small ϵ. Let K be a compact subset in Σ_{λ_0} such that $|\Sigma_{\lambda_0} \backslash K| < \delta/2$. Recall that δ is the measure of O for which the maximum principle for small domains in Lemma 3 holds. By the continuity of v_λ, there exists some $r > 0$ such that $v_\lambda > r$ in K. In the remaining $\Sigma_\lambda \backslash K$, we can check that v_λ satisfies

$$\begin{cases} \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 v_\lambda) - \gamma |Dv_\lambda| + c_\lambda(x) v_\lambda \leq 0 & \text{in } \Sigma_\lambda \backslash K, \\ v_\lambda \geq 0 & \text{on } \partial(\Sigma_\lambda \backslash K). \end{cases}$$

By the maximum principle for small domains again, $v_\lambda \geq 0$ in $\Sigma_\lambda \backslash K$ by selecting sufficiently small ϵ. Together with the fact that $v_\lambda \geq r$ in K, we infer that $v_\lambda \geq 0$ in Σ_λ. It contradicts the definition of λ_0.

Case 2: $\lambda_0 = -\frac{1}{2}$.

We also argue that the plane can be moved further, which indicates that $\lambda_0 = -\frac{1}{2}$ is impossible. Since O is symmetric with respect to the hyperplane $x_1 = 0$, then $0^{-1/2} = (-1, 0, \cdots, 0)$. We select a compact set K in $\Sigma_{-1/2}$ such that $|\Sigma_{\lambda_0} \backslash K| < \delta/2$. By the positivity and continuity of $v_{-1/2}$, there exists some $r > 0$ such that $v_{-1/2} > r$ in K. Without loss of generality, we may assume that $dist(K, \Sigma_{-1/2}) \geq r'$ for some $r' > 0$. We consider a small ball $\mathbb{B}_{r'/2}(e)$ centered at $e = (-1, 0, \cdots, 0)$ with radius $r'/2$. From the positivity of $v_{-1/2}$ again, we have, making r smaller if necessary,

$$v_{-1/2} > r/2 \text{ in } \partial\mathbb{B}_{r'/2}(e) \cap \bar{O}.$$

Let $\lambda = -1/2 + \epsilon$ for small $\epsilon > 0$. By the continuity of v_λ, we get

$$v_\lambda > r/4 \text{ in } (\partial\mathbb{B}_{r'/2}(e) \cap \bar{O}) \cup K.$$

For such small ϵ, $0^{-1/2+\epsilon}$ lies in $\mathbb{B}_{r'/2}(e) \cap \bar{O}$. We also know that $v_\lambda \geq 0$ on $\mathbb{B}_{r'/2}(e) \cap \partial O$. Therefore,

$$v_\lambda \geq 0 \text{ in } \partial(\mathbb{B}_{r'/2}(e) \cap O).$$

Choosing r' so small that Lemma 10 is valid, then

$$v_\lambda \geq 0 \text{ in } \mathbb{B}_{r'/2}(e) \cap O.$$

We consider the remaining set $\Sigma_\lambda \backslash (K \cup \mathbb{B}_{r'/2}(e))$. we can verify that v_λ satisfies

$$\begin{cases} \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 v_\lambda) - \gamma |Dv_\lambda| + c_\lambda(x) v_\lambda \leq 0 & \text{in } \Sigma_\lambda \backslash (K \cup \mathbb{B}_{r'/2}(e)), \\ v_\lambda \geq 0 & \text{on } \partial(\Sigma_\lambda \backslash (K \cup \mathbb{B}_{r'/2}(e))). \end{cases}$$

Therefore, for sufficiently small ϵ, the maximum principle of small domains implies that $v_\lambda \geq 0$ in $\Sigma_\lambda \backslash (K \cup \mathbb{B}_{r'/2})$. In conclusion, $v_\lambda \geq 0$ for $\lambda = -1/2 + \epsilon$. We arrive at a contradiction.

Case 3: $-1/2 < \lambda_0 < 0$.

We show that this critical position is also impossible. For the singular point 0^{λ_0}, we choose a ball $\mathbb{B}_{r'/2}(0^{\lambda_0})$ centered at 0^{λ_0} with radius $r'/2$. Let $\lambda = \lambda_0 + \epsilon$. For $\epsilon > 0$ small enough, 0^λ still lies in $\mathbb{B}_{r'/2}(0^{\lambda_0})$. By the continuity and positivity of v_{λ_0}, there exists some $r > 0$ such that $v_\lambda \geq r$ on $\partial \mathbb{B}_{r'/2}(0^{\lambda_0})$. Applying Lemma 10 for small value of $r'/2$, we infer that $v_\lambda \geq 0$ in $\mathbb{B}_{r'/2}(0^{\lambda_0})$. Similar argument as Case 1 and Case 2 could show that $v_\lambda \geq 0$ in $\Sigma_\lambda \backslash \{0\}$ for $\lambda = \lambda_0 + \epsilon$.

□

5. The Radial Symmetry for Viscosity Solutions of Fully Nonlinear Parabolic Equations

We consider the radial symmetry of fully nonlinear parabolic equation in this section. we first show that the difference of supersolution and subsolution of the parabolic equation satisfies an inequality involving Pucci extremal operator, which enables us to compare the value of u at x and its value at the reflection of x. The following lemma is non trivial since u is not of class C^2. The proof of the lemma below is inspired by the work in [**12**] and [**18**].

Lemma 11. *Let u_1, u_2 be a continuous subsolution and supersolution respectively in $\mathbb{R}^n \times (0,\ T]$ of*

$$\partial_t u - F(Du, D^2 u) - f(u) = 0. \tag{5.1}$$

Then $\tilde{w} = u_2 - u_1$ is a viscosity supersolution of

$$-\partial_t \tilde{w} + \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\tilde{w}) - \gamma|\nabla \tilde{w}| + c(x,t)\tilde{w} \leq 0, \tag{5.2}$$

where

$$c(x,t) = \begin{cases} \frac{f(u_1(x,t)) - f(u_2(x,t))}{u_1(x,t) - u_2(x,t)}, & \text{if } u_1(x,t) \neq u_2(x,t), \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

Proof. We consider $w = u_1 - u_2 = \tilde{w}$, then apply the property of $\mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 w) = -\mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\tilde{w})$ to verify (5.2). Let $\varphi \in C^2$ be a test function such that $w - \varphi$ has a local maximum at $(\tilde{x}, \tilde{t})$. Then there exists $r > 0$ such that, for all $(x,t) \in \overline{\mathbb{B}}_r(\tilde{x}) \times (\tilde{t} - r,\ \tilde{t}] \subset \mathbb{R}^n \times (0,\ T]$, $(w - \varphi)(x,t) < (w - \varphi)(\tilde{x}, \tilde{t})$. Define

$$\Phi_\epsilon(x,y,t) = u_1(x,t) - u_2(y,t) - \varphi(x,t) - \frac{|x-y|^2}{\epsilon^2}.$$

Let $(x_\epsilon, y_\epsilon, t_\epsilon)$ be the maximum point of $\Phi_\epsilon(x,y,t)$ in $\overline{\mathbb{B}}_r(\tilde{x}) \times \overline{\mathbb{B}}_r(\tilde{x}) \times (\tilde{t} - r,\ \tilde{t}]$. Standard argument shows that

$$\begin{cases} (i) : (x_\epsilon, y_\epsilon) \to (\tilde{x}, \tilde{x}), \\ (ii) : \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} \to 0 \end{cases} \tag{5.4}$$

as $\epsilon \to 0$. Let $\theta = \overline{\mathbb{B}}_r(\tilde{x})$ and

$$\psi_\epsilon(x,y,t) = \varphi(x,t) + \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2}.$$

The argument of Theorem 8.3 in [**8**] indicates that, for all $\alpha > 0$, there exist $X, Y \in S_n(\mathbb{R}^n)$ such that

$$\begin{cases} (i) : (a_\epsilon, D_x\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon), X) \in \overline{\mathcal{P}_\theta}^{2,+} u_1(x_\epsilon, t_\epsilon), \\ \qquad (b_\epsilon, D_y\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon), Y) \in \overline{\mathcal{P}_\theta}^{2,+}(-u_2)(y_\epsilon, t_\epsilon), \\ (ii) : -(1/\alpha + \|A\|)Id \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le A + \alpha A^2, \\ (iii) : a_\epsilon + b_\epsilon = \partial_t\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) = \partial_t\varphi(x_\epsilon, t_\epsilon), \end{cases} \tag{5.5}$$

where $A = D^2\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) = \begin{pmatrix} D_x^2\varphi(x_\epsilon, t_\epsilon) + \frac{2}{\epsilon^2}Id & -\frac{2}{\epsilon^2}Id \\ -\frac{2}{\epsilon^2}Id & \frac{2}{\epsilon^2}Id \end{pmatrix}$.

Furthermore, by the definition of ${\mathcal{P}_\theta}^{2,+}, {\mathcal{P}_\theta}^{2,-}$, we have

$$a_\epsilon - F(D_x\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon), X) - f(u_1(x_\epsilon, t_\epsilon)) \le 0, \tag{5.6}$$

$$-b_\epsilon - F(-D_y\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon), -Y) - f(u_2(y_\epsilon, t_\epsilon)) \ge 0. \tag{5.7}$$

Combining (iii) in (5.5), (5.6) and (5.7), we obtain

$$\partial_t\varphi(x_\epsilon, t_\epsilon) - F(D_x\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon), X) + F(-D_y\psi_\epsilon(x_\epsilon, y_\epsilon, t_\epsilon), -Y) - f(u_1(x_\epsilon, t_\epsilon)) + f(u_2(y_\epsilon, t_\epsilon)) \le 0. \tag{5.8}$$

Let $\alpha = \epsilon^2$. A similar argument to that in [**12**] leads to

$$X - D^2\varphi_\epsilon(x_\epsilon, t_\epsilon) + Y \le -C\epsilon^2 Y^2 + O(\epsilon) \tag{5.9}$$

for some $C > 0$. Then

$$\begin{aligned} (\partial_t\varphi - \mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\varphi) - f(u_1))(x_\epsilon, t_\epsilon) + f(u_2)(y_\epsilon, t_\epsilon) \;&-\; \gamma|D_x\psi + D_y\psi|(x_\epsilon, y_\epsilon, t_\epsilon) \\ &+\; C\epsilon^2\mathcal{M}^-_{\Lambda_1,\Lambda_2}(Y^2) + O(\epsilon) \le 0. \end{aligned}$$

Since $\mathcal{M}^-_{\Lambda_1,\Lambda_2}(Y^2) \ge 0$, letting $\epsilon \to 0$, then

$$(\partial_t\varphi - \mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\varphi) - \gamma|D_x\varphi| - f(u_1) + f(u_2))(\tilde{x}, \tilde{t}) \le 0.$$

By the mean value theorem,

$$(\partial_t\varphi - \mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2\varphi) - \gamma|D_x\varphi| - c(\tilde{x}, \tilde{t})(u_1 - u_2))(\tilde{x}, \tilde{t}) \le 0,$$

where $c(x,t)$ is in (5.3). Hence

$$\partial_t w - \mathcal{M}^+_{\Lambda_1,\Lambda_2}(D^2 w) - \gamma|D_x w| - c(x,t)w \le 0$$

for $(x,t) \in \mathbb{R}^n \times (0,T]$. Since $\tilde{w} = -w$,

$$-\partial_t\tilde{w} + \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2\tilde{w}) - \gamma|D_x\tilde{w}| + c(x,t)\tilde{w} \le 0.$$

The proof of the lemma is then fulfilled. □

We are ready to give the proof of Theorem 4.

Proof of Theorem 4. We adopt the moving plane method to prove the theorem. Define

$$\Sigma_\lambda = \{(x_1, x', t) \in \mathbb{R}^{n+1} | x_1 < \lambda, 0 < t \leq T\},$$

where $x' = \{x_2, \cdots, x_n\}$. Set

$$u_\lambda(x_1, x', t) = u(2\lambda - x_1, x', t) \text{ and } v_\lambda(x, t) = u_\lambda(x, t) - u(x, t).$$

Step 1: We start the plane from negative infinity. Since u_λ satisfies the same equation as u does by *(F2)*. Thanks to Lemma 11, we have

$$-\partial_t v_\lambda + \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 v_\lambda) - \gamma|\nabla v_\lambda| + c(x, t)v_\lambda \leq 0.$$

We may assume that $|c(x, t)| \leq c_0$ for some $c_0 > 0$, since $f(u)$ is locally Lipschitz. Let

$$\bar{v}_\lambda = \frac{v_\lambda}{e^{-(c_0+1)t}},$$

then $\bar{v}_\lambda$ satisfies

$$-\partial_t \bar{v}_\lambda + \mathcal{M}^-_{\Lambda_1,\Lambda_2}(D^2 \bar{v}_\lambda) - \gamma|\nabla \bar{v}_\lambda| + \tilde{c}(x, t)\bar{v}_\lambda \leq 0, \tag{5.10}$$

where $\tilde{c}(x, t) = c(x, t) - c_0 - 1$. Note that $\tilde{c}(x, t) < 0$. In order to prove that $v_\lambda \geq 0$ in Σ_λ, it is sufficient to show that $\bar{v}_\lambda \geq 0$ in Σ_λ. Suppose the contrary, that $\bar{v}_\lambda < 0$ somewhere in Σ_λ. Since

$$|u(x, t)| \to 0 \text{ uniformly as } |x| \to \infty,$$

then

$$v_\lambda(x, t) \geq -u(x, t)e^{(c_0+1)t} \to 0$$

as $|x| \to \infty$. Due to the fact that $\bar{v}_\lambda = 0$ on $\partial\Sigma_\lambda := \{(x_1, x', t)|x_1 = \lambda, 0 < t \leq T\}$ and the assumption of initial boundary condition $u_0(x)$, there exists some point $z^0 \in \Sigma_\lambda$ such that

$$\bar{v}_\lambda(z^0) = \min_{z\in\Sigma_\lambda} \bar{v}_\lambda(x, t) < 0.$$

By the strong maximum principle for fully nonlinear parabolic equations, we know it is a contradiction. Step 1 is then completed.

Step 2: Set

$$\lambda_0 := \sup\{\lambda < 0 | v_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for } -\infty < \mu < \lambda\}.$$

Our goal is to show that $\lambda_0 = 0$. Suppose that $\lambda_0 < 0$, then there exists sufficiently small $\epsilon > 0$ such that $\lambda_0 + \epsilon < 0$. We are going to prove that $v_\lambda \geq 0$ in Σ_λ for $\lambda = \lambda_0 + \epsilon$, which contradicts the definition of λ_0. If $v_{\lambda_0+\epsilon} < 0$ somewhere in $\Sigma_{\lambda_0+\epsilon}$, by the asymptotic behavior of u and the initial boundary condition, we know that the minimum point is achieved in the interior of $\Sigma_{\lambda_0+\epsilon}$. By the same argument as that in Step 1, we see it is impossible. Therefore, we confirm that $\lambda_0 = 0$, that is, u is nondecreasing in x_1 and $u(x_1, x', t) \leq u(-x_1, x', t)$ for $x_1 \leq 0$.

If the initial value u_0 is radial symmetry and nonincreasing in $|x|$. We move the plane from positive infinity to the left. By the same argument as above, we will reach at $\lambda_0 = 0$ again, which leads to the symmetry of the solution at $x_1 = 0$. By the rotation invariance of the equation, we obtain that u is radially symmetric with respect to $(0, t)$ for any fixed $t \in (0, T]$ and nonincreasing in $|x|$.

□

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