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Improved Moser-Trudinger Inequality Involving L^p Norm in n Dimensions

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Abstract

The paper is concerned about an improvement of Moser-Trudinger inequality involving L^p norm for a bounded domain in n dimensions. Let

$$\bar{\lambda}(\Omega) = \inf_{w \in H_0^{1,n}(\Omega), w \not\equiv 0} \frac{\|\nabla w\|_n^n}{\|w\|_p^n} \tag{0.1}$$

be the first eigenvalue associated with n-Laplacian. We obtain the following strengthened Moser-Trudinger inequality with blow-up analysis

$$\sup_{w \in H_0^{1,n}(\Omega), \|\nabla w\|_n = 1} \int_\Omega \exp\{\alpha_n |w|^{\frac{n}{n-1}} (1 + \alpha \|w\|_p^n)^{\frac{1}{n-1}}\} \, dx < \infty \tag{0.2}$$

for $0 \leq \alpha < \bar{\lambda}(\Omega)$ and $1 < p \leq n$, and the supremum is infinity for $\alpha \geq \bar{\lambda}(\Omega)$, where $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$. We also obtain the existence of the extremal functions for (0.2).

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1 Introduction

The sharp geometric inequalities and their extremal functions play an important role in analysis and geometry. The study of sharp constant for Moser-Trudinger inequality traces back to 1960s and 1970s. In 1971, Moser [18] elegantly sharpened the results of Phohozaev [19], Trudinger [23] and established the following so called Moser-Trudinger inequality

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$$\sup_{w\in H_0^{1,n}(\Omega),\|\nabla w\|_n=1}\int_\Omega \exp\{\alpha|w|^{\frac{n}{n-1}}\}\,dx<\infty \tag{1.1}$$

for any $\alpha\le\alpha_n$, where $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$, Ω is a smooth bounded domain in $\mathbb{R}^n$ and $H_0^{1,n}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$\|w\|_{H_0^{1,n}(\Omega)}=(\int_\Omega|\nabla w|^n\,dx+\int_\Omega|w|^n\,dx)^{\frac{1}{n}}.$$

We also use $\|\cdot\|_p$ to denote the L^p norm with respect to the Lebesgue measure. For any $\alpha>\alpha_n$, (1.1) is shown to be invalid by explicit test functions, i.e. there exists a sequence of $\{w_\epsilon\}$ in $H_0^{1,n}(\Omega)$ with $\|\nabla w_\epsilon\|_n=1$ such that

$$\int_\Omega \exp\{\alpha|w_\epsilon|^{\frac{n}{n-1}}\}\,dx\to\infty\quad\text{as}\quad\epsilon\to0.$$

On the other hand, for any fixed $w\in H_0^{1,n}(\Omega)$, it is also known that

$$\int_\Omega \exp\{\alpha|w|^{\frac{n}{n-1}}\}\,dx<\infty$$

for any α.

Recently, Lu and Yang in [16] considered an extension of the Moser-Trudinger inequality. Their work is motivated by Adimurthi-Druet [1] and Li [10] to some extent. Let

$$\lambda_p(\Omega)=\inf_{w\in H_0^{1,2}(\Omega),w\not\equiv0}\frac{\|\nabla w\|_2^2}{\|w\|_p^2}. \tag{1.2}$$

The main result in [16] is the following:

Theorem A Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and the eigenvalue $\lambda_p(\Omega)$ be as in (1.2). Then

(i) : For any $0\le\alpha<\lambda_p(\Omega)$,

$$\sup_{w\in H_0^{1,2}(\Omega),\|\nabla w\|_2=1}\int_\Omega \exp\{4\pi|w|^2(1+\alpha\|w\|_p^2)\}\,dx<\infty.$$

(ii) : For any $\alpha\ge\lambda_p(\Omega)$,

$$\sup_{w\in H_0^{1,2}(\Omega),\|\nabla w\|_2=1}\int_\Omega \exp\{4\pi|w|^2(1+\alpha\|w\|_p^2)\}\,dx=\infty.$$

The above theorem extends the main result of Adimurthi and Druet in [1], where the case $p=2$ is considered. In this paper, we consider the general n dimensional case of Theorem A. We define

$$\bar\lambda(\Omega)=\inf_{w\in H_0^{1,n}(\Omega),w\not\equiv0}\frac{\|\nabla w\|_n^n}{\|w\|_p^n}. \tag{1.3}$$

Adapting the idea in [11], [24] and [16], our first result is stated as:

Theorem 1.1 *Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Assume that the eigenvalue $\bar{\lambda}(\Omega)$ is as in (1.3) and $1 < p \le n$. Then*

(i) : *For any $0 \le \alpha < \bar{\lambda}(\Omega)$,*

$$\sup_{w \in H_0^{1,n}(\Omega), \|\nabla w\|_n = 1} \int_\Omega \exp\{\alpha_n |w|^{\frac{n}{n-1}} (1 + \alpha \|w\|_p^n)^{\frac{1}{n-1}}\}\, dx < \infty.$$

(ii) : *For any $\alpha \ge \bar{\lambda}(\Omega)$,*

$$\sup_{w \in H_0^{1,n}(\Omega), \|\nabla w\|_n = 1} \int_\Omega \exp\{\alpha_n |w|^{\frac{n}{n-1}} (1 + \alpha \|w\|_p^n)^{\frac{1}{n-1}}\}\, dx = \infty.$$

If $p = n$, the above result is established in [24]. Obviously, our result is more general. If $n = 2$, Theorem 1 is an extension of Theorem A. While $n = 2$ and $p = 2$, our theorem includes the main result in [1].

We sketch the idea of the proof of Theorem 1.1. The conclusion (ii) of Theorem 1.1 is verified based on the appropriate test functions. Without selecting test functions through Green function that results in tedious and complicated computations as [24], we choose the test functions involving the eigenfunction of the first eigenvalue in (1.3) inspired by [16]. Our test function is based on taking cut-off Green function inside and the eigenfunction outside. By computing the explicit test functions delicately, it naturally leads to the conclusion (ii) in Theorem 1.1. The proof of conclusion (i) in above theorem is relied on a blow-up analysis of sequences of solutions to elliptic PDEs with exponential growth in Ω. Integral estimate instead of pointwise estimate is studied. In order to handle the general case of $p \neq n$ and the n dimensions, more subtle estimates are involved than [16]. To be specific, the concentration point of the blow up sequence does not converge to the boundary of Ω in 2 dimensions by the moving plane method. However, the concentration point may approach to the boundary for n Laplacian (see section 3 for more details). Together with the classical Moser-Trudinger inequality and asymptotic estimate of the blow up sequence, we obtain the result in conclusion (i).

Another interesting and important investigation of the Moser-Trudiger inequality is about the existence. Carleson and Chang [3] first obtained the existence for (1.1) in the ball $\mathbb{B}$ in $\mathbb{R}^n$. Later Flucher [6] generalized the existence for any bounded domain in $\mathbb{R}^2$. Then Lin [14] proved the existence of (1.1) in any bounded smooth domain in $\mathbb{R}^n$. Li in [10],[11], Li and Liu [13] obtained the existence results for certain Moser-Trudinger inequalities on compact Riemannian manifolds with or without boundary. For our Moser-Trudinger inequality involving L^p norm in higher dimensions, we establish the following:

Theorem 1.2 *For any fixed $1 < p \le n$, there exists $w_\alpha \in H_0^{1,n}(\Omega)$ with $\|\nabla w_\alpha\|_n = 1$ such that*

$$\int_\Omega \exp\{\alpha_n |w_\alpha|^{\frac{n}{n-1}} (1 + \alpha \|w_\alpha\|_p^n)^{\frac{1}{n-1}}\}\, dx = \sup_{w \in H_0^{1,n}, \|\nabla w\|_n = 1} \int_\Omega \exp\{\alpha_n w^{\frac{n}{n-1}} (1 + \alpha \|w\|_p^n)^{\frac{1}{n-1}}\}\, dx,$$

(i) : *if $n = 2$ and α is sufficient small, or*

(ii) : *if $n \ge 3$ and $0 \le \alpha < \bar{\lambda}(\Omega)$.*

Concerning about the existence in higher dimensions $n \ge 3$, it is interesting that the existence of the improved Moser-Trudinger inequality with L^p norm holds for the whole range of α derived in conclusion of Theorem 1.2. While in the case of $n = 2$, the extremal functions could only be found for small α. The strategy in establishing Theorem 1.2 is the application of a contradiction argument. The idea of proving our existence results is inspired by [10]. On one hand, We find a upper bound for our Moser-Trudinger function with L^p norm from the Carleson-Chang Theorem under the assumption that the blow-up sequence exists. On the other hand, A sequence of test functions can be constructed to achieve the exactly same lower bound. This contradiction yields the fact that no blow-up occurs. Then Theorem 1.2 follows. Also in the n dimensions and for general case $1 < p \leq n$, more intricate calculation arises. Instead of choosing the general test functions in [24],[12], we provide some of kind of concrete test functions to attain the lower bound.

The blow-up analysis is widely employed in the paper. This approach for elliptic equations related to the classical Moser-Trudinger inequality was initiated in [3],[2], [10] and [1]. Similar approaches and relevant existence results are also implemented in [12], [17] and references therein.

We also remark that sharp Moser-Trudinger inequalities in the Hesienberg group and Adams inequalities in high order Sobolev spaces have been established. We refer the reader to [4], [5], [7], [8], [9], [20], etc., just to name a few.

The rest of the paper is arranged as follows. In Section 2, we construct test functions to show conclusion (ii) of Theorem 1.1. In Section 3, we consider the relevant Euler-Lagrange equation for the maximizers of the subcritical Moser-Trudinger function with L^p norm and investigate the asymptotic behavior of the maximizers through blow-up analysis. Then it leads to the conclusion (i) of Theorem 1.1. Section 4 is devote to the existence of the extremal functions of the improved Moser-Trudinger inequality, i.e. Theorem 1.2. The letter C denotes a positive constant, which may change from line to line.

2 The Test Functions Argument

In this section, we prove the conclusion (ii) of Theorem 1.1. We will build explicit test functions to show the unboundedness of Moser-Trudinger function under suitable large parameter. We first verify that the eigenvalue and eigenfunction of (1.3) is achievable.

Lemma 2.1 *For any $p > 1$, $\bar{\lambda}(\Omega) > 0$ in (1.3) is attained by the eigenvalue function $\varphi \in H_0^{1,n}(\Omega) \cap C^1(\Omega)$ satisfying*

$$\begin{cases} -\triangle_n \varphi = \bar{\lambda}(\Omega) \|\varphi\|_p^{n-p} \varphi^{p-1} & \text{in } \Omega, \\ \|\nabla\varphi\|_n = 1, \quad \varphi > 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Proof. The proof is a direct consequence of variation method. We present it here for completeness. We select a sequence of $\{w_k\}$ such that $\|w_k\|_p = 1$ and $\|\nabla w_k\|_n^n \to \bar{\lambda}(\Omega)$ as $k \to \infty$. Obviously w_k is bounded in $H^{1,n}(\Omega)$. We may assume that there exists a subsequence of w_k such that

$$w_k \rightharpoonup w_0 \quad \text{weakly in } H^{1,n}(\Omega),$$

$$w_k \to w_0 \quad \text{strongly in } L^p(\Omega).$$

It follows that $\|w_0\|_p = 1$. We further infer that

$$\int_\Omega |\nabla w_0|^n \, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla w_k|^n \, dx \to \bar{\lambda}(\Omega).$$

Thus, $\bar{\lambda}(\Omega) = \int_\Omega |\nabla w_0|^n \, dx > 0$. Since $\|\nabla |w_0|\|_n^n = \|\nabla w_0\|_n^n$, we may assume that $|w_0|$ achieves

$$\inf_{w \in H_0^{1,n}(\Omega), w \not\equiv 0} \frac{\|\nabla w\|_n^n}{\|w\|_p^n}.$$

Let $\varphi = \frac{|w_0|}{\|\nabla |w_0|\|_n}$. φ attains the above infimum and satisfies the Euler-Lagrange equation (2.1). The positiveness of φ follows from the strong maximum principle.

We are ready to give the proof the second conclusion in Theorem 1.1.

Proof. [Proof of the Conclusion (ii) in Theorem 1.1] Since the Moser-Trudinger inequality is invariant under translation, we may assume that $0 \in \Omega$ and $\mathbb{B}_1 \subset \Omega$. We fix some $x_\delta \in \mathbb{B}_1$ such that $|x_\delta| = \delta$. Choosing t_ϵ such that $t_\epsilon^n \log \frac{1}{\epsilon} \to \infty$ and $t_\epsilon^{n+1} \log \frac{1}{\epsilon} \to 0$. Such t_ϵ is attainable. For instance, $t_\epsilon = (\log(\frac{1}{\epsilon}))^{\frac{-2}{2n+1}}$. Set

$$\varphi_\epsilon(x) = \begin{cases} (\frac{n}{\alpha_n} \log \frac{1}{\epsilon})^{\frac{n-1}{n}}, & |x| \le \epsilon, \\ \frac{(\frac{n}{\alpha_n} \log \frac{1}{\epsilon})^{\frac{n-1}{n}} (\log\delta - \log|x|) - t_\epsilon \varphi(x_\delta)(\log\epsilon - \log|x|)}{\log\delta - \log\epsilon}, & \epsilon \le |x| \le \delta, \\ t_\epsilon[\varphi(x_\delta) + \theta(x)(\varphi(x) - \varphi(x_\delta))], & |x| > \delta. \end{cases}$$

In above definition of $\varphi_\epsilon(x)$, φ is the eigenvalue function in Lemma 2.1, $\theta(x) \in C^2(\Omega)$ is a cut-off function satisfying $|\nabla\theta| \le C/\delta$ and

$$\theta(x) = \begin{cases} 0, & |x| \le \delta, \\ \theta \in (0,\, 1), & \delta < |x| < 2\delta, \\ 1, & |x| \ge 2\delta. \end{cases} \tag{2.2}$$

Let $\delta = 1/(t_\epsilon(\log \frac{1}{\epsilon})^{\frac{1}{n}})^p$. It is not hard to see that $\epsilon < \delta$ if ϵ is small enough. We obtain that

$$\begin{aligned} \int_{\epsilon\le|x|\le\delta} |\nabla\varphi_\epsilon(x)|^n\, dx &= \int_{\epsilon\le|x|\le\delta} \frac{|-(\frac{n}{\alpha_n}\log\frac{1}{\epsilon})^{\frac{n-1}{n}} + t_\epsilon\varphi(x_\delta)|^n}{|x|^n(\log\delta - \log\epsilon)^n}\, dx \\ &= \frac{\omega_{n-1}|-(\frac{n}{\alpha_n}\log\frac{1}{\epsilon})^{\frac{n-1}{n}} + t_\epsilon\varphi(x_\delta)|^n}{(\log\delta-\log\epsilon)^{n-1}} \\ &= 1 - n\omega_{n-1}^{\frac{1}{n}}(\log\frac{1}{\epsilon})^{-\frac{n-1}{n}} t_\epsilon\varphi(x_\delta)(1 + o_\epsilon(1)), \end{aligned}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. We also have

$$\begin{aligned} \int_{\delta\le|x|\le2\delta} |\nabla\varphi_\epsilon(x)|^n\, dx &= t_\epsilon^n \int_{\delta\le|x|\le2\delta} |\theta(x)\nabla\varphi(x) + \nabla\theta(x)(\varphi(x) - \varphi(x_\delta))|^n\, dx \\ &= t_\epsilon^n O(\delta^n) \end{aligned}$$

and

$$\begin{aligned} \int_{|x|\ge2\delta} |\nabla\varphi_\epsilon(x)|^n\, dx &= t_\epsilon^n \int_{|x|\ge2\delta} |\nabla\varphi(x)|^n\, dx \\ &= t_\epsilon^n(1 + O(\delta^n)). \end{aligned}$$

Summing the above integral estimates for $|\nabla\varphi_\epsilon(x)|^n$ up, we get

$$\int_\Omega |\nabla\varphi_\epsilon(x)|^n\, dx = 1 - n\omega_{n-1}^{\frac{1}{n}}(\log\frac{1}{\epsilon})^{-\frac{n-1}{n}} t_\epsilon\varphi(x_\delta)(1 + o_\epsilon(1)) + t_\epsilon^n(1 + O(\delta^n)).$$

Then

$$\|\nabla\varphi_\epsilon\|_n^{-\frac{n}{n-1}} = 1 + \frac{n}{n-1}\omega_{n-1}^{\frac{1}{n}}(\log\frac{1}{\epsilon})^{-\frac{n-1}{n}} t_\epsilon\varphi(x_\delta)(1 + o_\epsilon(1)) - \frac{1}{n-1} t_\epsilon^n(1 + O(\delta^n)). \tag{2.3}$$

Set $v_\epsilon = \frac{\varphi_\epsilon}{\|\nabla\varphi_\epsilon\|_n}$, then $\|\nabla v_\epsilon\|_n = 1$. Furthermore,

$$\begin{aligned}
\bar{\lambda}(\Omega)\|v_\epsilon\|_p^n &\geq \tfrac{\bar{\lambda}(\Omega)t_\epsilon^n}{\|\nabla\varphi_\epsilon\|_n^n}\big(\textstyle\int_{|x|\geq 2\delta}|\varphi(x)|^p\,dx\big)^{\frac{n}{p}}\\
&\geq \bar{\lambda}(\Omega)t_\epsilon^n[\|\varphi\|_p^n + O(\delta^{\frac{n^2}{p}})]\{1 + n\omega_{n-1}^{\frac{1}{n}}(\log\tfrac{1}{\epsilon})^{-\frac{n-1}{n}}t_\epsilon\varphi(x_\delta)(1+o_\epsilon(1))\\
&\quad -t_\epsilon^n(1+O(\delta^n))\}\\
&= t_\epsilon^n(\bar{\lambda}(\Omega)\|\varphi\|_p^n + O(\delta^{\frac{n^2}{p}}))(1+O(t_\epsilon^n))\\
&= t_\epsilon^n(1 + O(t_\epsilon^n) + O(\delta^{\frac{n^2}{p}})),
\end{aligned}$$

where $\bar{\lambda}(\Omega)\|\varphi_0\|_p^n = 1$ in Lemma 2.1 is used.

Next we establish the integral estimates on the domain of $\{x \in \Omega : |x| < \epsilon\}$. We have

$$\begin{aligned}
\alpha_n(1+\bar{\lambda}\|v_\epsilon\|_p^n)^{\frac{1}{n-1}}|v_\epsilon|^{\frac{n}{n-1}} &\geq n\log\tfrac{1}{\epsilon}(1+\bar{\lambda}(\Omega)\|v_\epsilon\|_p^n)^{\frac{1}{n-1}}\|\nabla\varphi_\epsilon\|_n^{\frac{-n}{n-1}}\\
&= n\log\tfrac{1}{\epsilon}(1 + t_\epsilon^n(1+O(t_\epsilon^n)+O(\delta^{\frac{n^2}{p}}))^{\frac{1}{n-1}}\\
&\quad\cdot(1+\tfrac{n}{n-1}\omega_{n-1}^{\frac{1}{n}}(\log\tfrac{1}{\epsilon})^{-\frac{n-1}{n}}t_\epsilon\varphi(x_\delta)(1+o(1))\\
&\quad -\tfrac{1}{n-1}t_\epsilon^n(1+O(\delta^n)))\\
&= n\log\tfrac{1}{\epsilon} + \tfrac{n^2}{n-1}\omega_{n-1}^{\frac{1}{n}}(\log\tfrac{1}{\epsilon})^{\frac{1}{n}}t_\epsilon\varphi(x_\delta)(1+o_\epsilon(1))\\
&\quad -\tfrac{n}{n-1}\log\tfrac{1}{\epsilon}t_\epsilon^n(1+O(\delta^n)) + \tfrac{n}{n-1}\log\tfrac{1}{\epsilon}t_\epsilon^n(1+O(t_\epsilon^n)\\
&\quad +O(\delta^{\frac{n^2}{p}})) + o_\epsilon(1)\\
&= n\log\tfrac{1}{\epsilon} + \tfrac{n^2}{n-1}\omega_{n-1}^{\frac{1}{n}}(\log\tfrac{1}{\epsilon})^{\frac{1}{n}}t_\epsilon\varphi(0)(1+o_\epsilon(1))\\
&\quad +\tfrac{n}{n-1}\log\tfrac{1}{\epsilon}t_\epsilon^nO(\delta^{\frac{n^2}{p}}) + \tfrac{n}{n-1}\log\tfrac{1}{\epsilon}t_\epsilon^nO(\delta^n) + o_\epsilon(1),
\end{aligned}$$

where the fact that $\varphi(x_\delta) = \varphi(0) + o_\epsilon(1)$ is applied. Note that for $p > 1$,

$$\log\frac{1}{\epsilon}t_\epsilon^nO(\delta^n) = o_\epsilon(1),$$

$$\log\frac{1}{\epsilon}t_\epsilon^nO(\delta^{\frac{n^2}{p}}) = o_\epsilon(1).$$

Considering the above estimates, we deduce that

$$\begin{aligned}
\textstyle\int_\Omega \exp\{\alpha_n(1+\bar{\lambda}\|v_\epsilon\|_p^n)^{\frac{1}{n-1}}|v_\epsilon|^{\frac{n}{n-1}}\} &\geq C\exp\{\tfrac{n^2}{n-1}w_{n-1}^{\frac{1}{n}}(\log\tfrac{1}{\epsilon})^{\frac{1}{n}}t_\epsilon\varphi_0(0)(1+o(1))\}\\
&\to \infty
\end{aligned}$$

as $\epsilon \to 0$, since $\varphi(0) > 0$ and $(\log\frac{1}{\epsilon})^{\frac{1}{n}}t_\epsilon \to \infty$. Here C is a positive constant independent of ϵ. The conclusion (ii) in Theorem 1 is completed.

3 Extremal of Subcritical Functions

In this section, we establish the conclusion (i) in Theorem 1.1. We first introduce some notations. Let

$$I_\beta^\alpha(w) = \int_\Omega \exp\{\beta(1+\alpha\|w\|_p^n)^{\frac{1}{n-1}}|w|^{\frac{n}{n-1}}\}\,dx$$

and

$$\mathcal{H} = \{w \in H_0^{1,n}(\Omega) | \|\nabla w\|_n = 1\}.$$

We also denote $\bar{\lambda}(\Omega)$ by $\bar{\lambda}$ for simplicity. We first present a technical lemma contributed by P. L. Lions [15].

Lemma 3.1 *Assume that $w_\epsilon \in H_0^{1,n}(\Omega)$, $\|\nabla w_\epsilon\|_n = 1$ and $w_\epsilon \rightharpoonup w_0$ weakly in $H_0^{1,n}(\Omega)$. Then, for any $q < (1 - \|\nabla w_0\|_n^n)^{\frac{-1}{n-1}}$,*

$$\limsup_{\epsilon \to 0} \int_\Omega \exp\{\alpha_n q |w_\epsilon|^{\frac{n}{n-1}}\} \, dx < \infty.$$

Clearly, in the case of $w_0 \not\equiv 0$, Lions' result provides more information than (1.1). We begin with the following existence lemma of the maximizer of the subcritical Moser-Trudinger function.

Lemma 3.2 *For any small ϵ and $0 \le \alpha < \bar{\lambda}$, there exists an extremal function $w_\epsilon \in C^1(\bar{\Omega}) \cap \mathcal{H}$ such that*

$$I^\alpha_{\alpha_n - \epsilon}(w_\epsilon) = \sup_{w \in \mathcal{H}} I^\alpha_{\alpha_n - \epsilon}(w).$$

Proof. For any $\epsilon > 0$, there exists a sequence of $\{w_i\} \in \mathcal{H}$ such that

$$\lim_{i \to \infty} I^\alpha_{\alpha_n - \epsilon}(w_i) = \sup_{w \in \mathcal{H}} I^\alpha_{\alpha_n - \epsilon}(w).$$

Since w_i is bounded in $H_0^{1,n}(\Omega)$, there exists a subsequence of w_i (We do not distinguish subsequence and sequence in the paper. It could be recognized from the context) such that

$$w_i \rightharpoonup w_\epsilon \text{ weakly in } H_0^{1,n}(\Omega),$$
$$w_i \to w_\epsilon \text{ strongly in } L^q(\Omega),$$
$$w_i \to w_\epsilon \ a.e. \text{ in } \Omega$$

for any $1 < q < \infty$ as $i \to \infty$. Hence

$$g_i := \exp\{(\alpha_n - \epsilon)(1 + \alpha \|w_i\|_p^n)^{\frac{1}{n-1}} |w_i|^{\frac{n}{n-1}}\} \to g_\epsilon := \exp\{(\alpha_n - \epsilon)(1 + \alpha \|w_\epsilon\|_p^n)^{\frac{1}{n-1}} |w_\epsilon|^{\frac{n}{n-1}}\}$$

a.e. in Ω. Thanks to Lions' result (Lemma 3.1), for any $q < 1/(1 - \|\nabla w_\epsilon\|_n^n)^{\frac{-1}{n-1}}$,

$$\limsup_{i \to \infty} \int_\Omega \exp\{\alpha_n q w_i^{\frac{n}{n-1}}\} \, dx < \infty.$$

Due to Lemma 2.1,

$$1 + \alpha \|w_\epsilon\|_p^n < \frac{1}{1 - \|\nabla w_\epsilon\|_n^n}$$

for $0 \le \alpha < \bar{\lambda}$. Thus, g_i is bounded in $L^s(\Omega)$ for some $s > 1$. Since $g_i \to g_\epsilon$ $a.e$ in Ω , we infer that $g_i \to g_\epsilon$ strongly in $L^1(\Omega)$ as $i \to \infty$. Therefore, the extremal function is attained for the case of $\alpha_n - \epsilon$ and $\|\nabla w_\epsilon\|_n = 1$.

Lemma 3.3 *$\forall \alpha, 0 \le \alpha < \bar{\lambda}$,*

$$\lim_{\epsilon \to 0} I^\alpha_{\alpha_n - \epsilon}(w_\epsilon) = \sup_{w \in \mathcal{H}} I^\alpha_{\alpha_n}(w).$$

Proof. The proof is similar to [24]. The interested reader may refer to [24] for the details.

Thanks to Lemma 3.3, in order to prove the conclusion (i) in Theorem 1.1, we focus on the extremal function w_ϵ. From the explicit form of the improved Moser-Trudinger inequality with L^p norm, we only consider the nonnegative w_ϵ. The Euler-Lagrange equation for $w_\epsilon \in H_0^{1,n}(\Omega) \cap C^1(\Omega)$ of $I_{\alpha_n-\epsilon}^{\alpha}(w_\epsilon)$ is

$$-\triangle_n w_\epsilon = \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} + \gamma_\epsilon \|w_\epsilon\|_p^{n-p} w_\epsilon^{p-1}, \tag{3.1}$$

where

$$\begin{cases} w_\epsilon \in H_0^{1,n}(\Omega), \quad \|\nabla w_\epsilon\|_n = 1, \\ \alpha_\epsilon = (\alpha_n - \epsilon)(1 + \alpha\|w_\epsilon\|_p^n)^{\frac{1}{n-1}}, \\ \beta_\epsilon = (1 + \alpha\|w_\epsilon\|_p^n)/(1 + 2\alpha\|w_\epsilon\|_p^n), \\ \gamma_\epsilon = \alpha/(1 + 2\alpha\|w_\epsilon\|_p^n), \\ \lambda_\epsilon = \int_\Omega |w_\epsilon|^{\frac{n}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\, dx. \end{cases} \tag{3.2}$$

Let $m_\epsilon = w_\epsilon(x_\epsilon) = \max_\Omega w_\epsilon(x)$. We may assume that $m_\epsilon \to \infty$ as $\epsilon \to \infty$. Otherwise, if m_ϵ is bounded, by applying elliptic estimate e.g. [22] to (3.1), The conclusion (i) in Theorem 1 and Theorem 2 follow directly. Since $\bar{\Omega}$ is a compact set in $\mathbb{R}^n$, $x_\epsilon \to z$ for some $z \in \bar{\Omega}$ as $\epsilon \to 0$. Two cases may occur if the blow-up sequence exists, that is, the concentration point z lies in the interior of Ω or z lies on $\partial\Omega$. We are going to analyze the asymptotic behavior of w_ϵ in those cases, respectively.

Case 1. z lies in the interior of Ω.

Lemma 3.4 *If $m_\epsilon \to \infty$, then $w_\epsilon \rightharpoonup 0$ weakly in $H^{1,n}(\Omega)$, $w_\epsilon \to 0$ strongly in $L^q(\Omega)$ for any $q > 1$ and $|\nabla w_\epsilon|^n\, dx \to \delta_z$ in sense of measure as $\epsilon \to 0$, where δ_z is the Dirac measure at z.*

Proof. Since w_ϵ is bounded in $H_0^{1,n}(\Omega)$, we may assume that

$$\begin{aligned} &w_\epsilon \rightharpoonup w_0 \ \text{ weakly in } \ H_0^{1,n}(\Omega), \\ &w_\epsilon \to w_0 \ \text{ strongly in } \ L^q(\Omega) \end{aligned}$$

for any $q > 1$ as $\epsilon \to 0$.

Suppose $w_0 \not\equiv 0$. For any $0 \le \alpha < \bar{\lambda}(\Omega)$, we have

$$1 + \alpha\|w_\epsilon\|_p^n \to 1 + \alpha\|w_0\|_p^n < \frac{1}{1 - \|\nabla w_0\|_n^n}.$$

Thanks to Lions' result (Lemma 3.1), $\exp\{\alpha_\epsilon w^{\frac{n}{n-1}}\}$ is bounded in $L^s(\Omega)$ for some $s > 1$. For sufficient small ϵ, using Hölder's inequality,

$$-\triangle_n w_\epsilon \in L^{s_0}(\Omega)$$

for some $s_0 > 1$. Classical elliptic estimate implies that w_ϵ is bounded in a neighborhood of z. It contradicts the assumption that $m_\epsilon \to \infty$. Moreover,

$$\alpha_\epsilon \to \alpha_n, \ \beta_\epsilon \to 1, \ \text{ and } \ \gamma_\epsilon \to \alpha$$

as $\epsilon \to 0$.

Assume that $|\nabla w_\epsilon|^n dx \rightharpoonup \mu$ in the sense of measure as $\epsilon \to 0$. If $\mu \neq \delta_z$, we claims that there exists a cut-off function $\phi \in C_0^1(\Omega)$, which is supported in $\mathbb{B}_r(z) \Subset \Omega$ for some $r > 0$ with $0 < \phi(x) < 1$ in $\mathbb{B}_r(z)\backslash\mathbb{B}_{r/2}(z)$ and $\phi(x) = 1$ in $\mathbb{B}_{r/2}(z)$ satisfying

$$\int_{\mathbb{B}_r(z)} \phi|\nabla w_\epsilon|^n \, dx \leq 1 - \eta$$

for some $\eta > 0$ and small enough ϵ. We prove the claim by contradiction. There exist sequences of $\eta_i \to 0$ and $r_i \to 0$ as $i \to \infty$ such that

$$\int_{\mathbb{B}_{r_i}(z)} \phi_i|\nabla w_\epsilon|^n \, dx > 1 - \eta_i$$

for every $\phi_i(x) \in C_0^1(\mathbb{B}_r(z))$ and $\phi_i(x) = 1$ in $\mathbb{B}_{r_i/2}(z)$. Then

$$\int_{\mathbb{B}_{r_i/2}(z)} |\nabla w_\epsilon|^n \, dx > 1 - \eta_i. \tag{3.3}$$

Taking $i \to \infty$, the left hand side of (3.3) converges to 0. However, $1 - \eta_i \to 1$. This contradiction leads to the claim. Since $w_\epsilon \to 0$ strongly in $L^q(\Omega)$ for any $q > 1$, we may assume that

$$\int_{\mathbb{B}_r(z)} |\nabla(\phi w_\epsilon)|^n \, dx \leq 1 - \eta$$

provided ϵ is sufficient small. By the classical Moser-Trudinger inequality, $\exp\{\alpha_n w_\epsilon^{\frac{n}{n-1}}\}$ is bounded in $L^s(\mathbb{B}_{r_0}(z))$ for some $s > 1$ and $0 < r_0 < r$. Applying the elliptic estimates, w_ϵ is bounded in $\mathbb{B}_{r_0/2}(z)$, which contradicts the fact that $m_\epsilon \to \infty$ again. Therefore, $|\nabla w_\epsilon|^n \, dx \rightharpoonup \delta_z$ as $\epsilon \to 0$.

Let

$$\begin{cases} r_\epsilon = \lambda_\epsilon^{\frac{1}{n}} \beta_\epsilon^{\frac{-1}{n}} m_\epsilon^{\frac{-1}{n-1}} \exp\{\frac{-\alpha_\epsilon m_\epsilon^{\frac{n}{n-1}}}{n}\}, \\ \phi_\epsilon = \frac{1}{m_\epsilon} w_\epsilon(x_\epsilon + r_\epsilon x), \\ \psi_\epsilon(x) = m_\epsilon^{\frac{1}{n-1}}(w_\epsilon(x_\epsilon + r_\epsilon x) - m_\epsilon). \end{cases} \tag{3.4}$$

Note that ϕ_ϵ and ψ_ϵ are defined in $\Omega_\epsilon := \{x \in \mathbb{R}^n : x_\epsilon + r_\epsilon x \in \Omega\}$. Following from the Euler-Lagrange equation (3.1), $\phi_\epsilon, \psi_\epsilon$ satisfy

$$-\triangle_n \phi_\epsilon(x) = m_\epsilon^{-n} \phi_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon(w_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - m_\epsilon^{\frac{n}{n-1}})\} + m_\epsilon^{p-n} r_\epsilon^n \gamma_\epsilon \|w_\epsilon\|_p^{n-p} \phi_\epsilon^{p-1} \tag{3.5}$$

and

$$-\triangle_n \psi_\epsilon(x) = \phi_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon(w_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - m_\epsilon^{\frac{n}{n-1}})\} + m_\epsilon^p r_\epsilon^n \gamma_\epsilon \|w_\epsilon\|_p^{n-p} \phi_\epsilon^{p-1}, \tag{3.6}$$

respectively.

Lemma 3.5 *Fixed any* $0 < \delta < \alpha_n/2$, *we have* $r_\epsilon^n \exp\{\delta m_\epsilon^{\frac{n}{n-1}}\} \to 0$ *as* $\epsilon \to 0$.

Proof. By the expression of r_ϵ in (3.4) and λ_ϵ in (3.2),

$$\begin{aligned} r_\epsilon^n \exp\{\delta m_\epsilon^{\frac{n}{n-1}}\} &= \lambda_\epsilon \beta_\epsilon^{-1} m_\epsilon^{\frac{-n}{n-1}} \exp\{-\alpha_\epsilon m_\epsilon^{\frac{n}{n-1}}\} \exp\{\delta m_\epsilon^{\frac{n}{n-1}}\} \\ &= \beta_\epsilon^{-1} m_\epsilon^{\frac{-n}{n-1}} \exp\{(\delta - \alpha_\epsilon) m_\epsilon^{\frac{n}{n-1}}\} \int_\Omega w_\epsilon^{\frac{n}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} \\ &\leq \beta_\epsilon^{-1} m_\epsilon^{\frac{-n}{n-1}} \exp\{(2\delta - \alpha_\epsilon) m_\epsilon^{\frac{n}{n-1}}\} \int_\Omega w_\epsilon^{\frac{n}{n-1}} \exp\{(\alpha_\epsilon - \delta) w_\epsilon^{\frac{n}{n-1}}\} \\ &\leq C\beta_\epsilon^{-1} m_\epsilon^{\frac{-n}{n-1}} \exp\{(2\delta - \alpha_\epsilon) m_\epsilon^{\frac{n}{n-1}}\} \\ &\to 0 \end{aligned}$$

as $\epsilon \to 0$. In above, we have applied Hölder inequality, the classical Moser-Trudinger inequality, and the fact that $\beta_\epsilon \to 1$, $m_\epsilon \to \infty$, $\alpha_\epsilon \to \alpha_n$ as $\epsilon \to 0$.

By the fact that $\|\phi_\epsilon\|_\infty = 1$ and Lemma 3.5, the right hand side of (3.5) vanishes as $\epsilon \to 0$. Applying the classical estimates [22],

$$\phi_\epsilon \to \phi \ \text{ in } C^1_{loc}(\mathbb{R}^n), \quad \text{as } \epsilon \to 0$$

and

$$-\triangle_n \phi = 0 \ \text{ in } \mathbb{R}^n.$$

Since $\phi_\epsilon(0) = 1$, standard Liouville-type theorem yields that $\phi(x) \equiv 1$ in $\mathbb{R}^n$.

Now we investigate the behavior of ψ_ϵ. For $x \in \mathbb{B}_R(0)$,

$$\begin{aligned} w_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - m_\epsilon^{\frac{n}{n-1}} &= m_\epsilon^{\frac{n}{n-1}}(\phi_\epsilon^{\frac{n}{n-1}} - 1) \\ &= m_\epsilon^{\frac{n}{n-1}}(\tfrac{n}{n-1}(\phi_\epsilon - 1) + O((\phi_\epsilon - 1)^2)) \\ &= \tfrac{n}{n-1}\psi_\epsilon(x) + o_\epsilon(\psi_\epsilon(x)). \end{aligned}$$

Because of the negativeness of ψ_ϵ, using the local estimates of n-Laplacian and Lemma 3.5, we obtain that ψ_ϵ is bounded in $L^\infty(\mathbb{B}_{R/2})$ in (3.6). Furthermore, ψ_ϵ is bounded in $C^{1,\mu}(\mathbb{B}_{R/4})$ for some $0 < \mu < 1$. Due to Arzelá-Ascoli Theorem, there exists some ψ such that $\psi_\epsilon \to \psi$ in $C^1(\mathbb{B}_{R/6})$. Let $R \to \infty$, we get $\psi_\epsilon \to \psi$ in $C^1_{loc}(\mathbb{R}^n)$ as $\epsilon \to 0$. Moreover,

$$\begin{aligned} \int_{\mathbb{B}_{R/6}(0)} \exp\{\tfrac{n}{n-1}\alpha_n\psi\}\,dx &\le \liminf_{\epsilon\to 0} \int_{\mathbb{B}_{R/6}(0)} \exp\{\alpha_\epsilon(w_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - m_\epsilon^{\frac{n}{n-1}})\}\,dx \\ &= \liminf_{\epsilon\to 0} \int_{\mathbb{B}_{Rr_\epsilon/6}(x_\epsilon)} \exp\{\alpha_\epsilon(w_\epsilon^{\frac{n}{n-1}}(x) - m_\epsilon^{\frac{n}{n-1}})\} r_\epsilon^{-n}\,dx \\ &= \liminf_{\epsilon\to 0}(1 + o_\epsilon(1))\lambda_\epsilon^{-1} \int_{\mathbb{B}_{Rr_\epsilon/6}(x_\epsilon)} w_\epsilon^{\frac{n}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx \\ &\le 1. \end{aligned}$$

Hence, ψ satisfies

$$\begin{cases} -\triangle_n \psi = \exp\{\tfrac{n}{n-1}\alpha_n\psi\}, \\ \psi(0) = \sup_{\mathbb{R}^n}\psi = 0, \\ \int_{\mathbb{R}^n} \exp\{\tfrac{n}{n-1}\alpha_n\psi\} \le 1. \end{cases} \tag{3.7}$$

By solving a corresponding ODE,

$$\psi(x) = -\frac{n-1}{\alpha_n}\log(1 + (\frac{\omega_{n-1}}{n})^{\frac{1}{n-1}}|x|^{\frac{n}{n-1}}). \tag{3.8}$$

The interested readers may refer to [11] for similar arguments. The above reasoning is summarized in the following lemma.

Lemma 3.6 *$\phi_\epsilon \to 1$ and $\psi_\epsilon \to \psi$ in $C^1_{loc}(\mathbb{R}^n)$ as $\epsilon \to 0$, where ψ is in (3.8).*

Define $w_{\epsilon,d} = \min(w_\epsilon, dm_\epsilon)$, where $0 < d < 1$.

Lemma 3.7 *We have*

$$\lim_{\epsilon\to 0}\int_\Omega |\nabla w_{\epsilon,d}|^n\,dx = d \tag{3.9}$$

and

$$\lim_{\epsilon\to 0}\int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx = 1 - d. \tag{3.10}$$

Proof. Since $\phi_\epsilon \to 1$ in $C^1_{loc}(\mathbb{R}^n)$, then $w_{\epsilon,d} \geq dm_\epsilon$ in $\mathbb{B}_{Rr_\epsilon}(x_\epsilon)$. On one hand, from equation (3.1),

$$\begin{aligned}\int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx &= -\int_\Omega (w_\epsilon - dm_\epsilon)^+ \triangle_n w_\epsilon\,dx \\ &= \int_\Omega (w_\epsilon - dm_\epsilon)^+ \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} \\ &\quad + (w_\epsilon - dm_\epsilon)^+ \gamma_\epsilon \|w_\epsilon\|_p^{n-p} w_\epsilon^{p-1}\,dx \\ &\geq \int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} (w_\epsilon - dm_\epsilon) \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} \\ &\quad + (w_\epsilon - dm_\epsilon) \gamma_\epsilon \|w_\epsilon\|_p^{n-p} w_\epsilon^{p-1}\,dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} (w_\epsilon - dm_\epsilon) \gamma_\epsilon \|w_\epsilon\|_p^{n-p} w_\epsilon^{p-1}\,dx &\leq \gamma_\epsilon \|w_\epsilon\|_p^{n-p} \|w_\epsilon\|_p^p \\ &\to 0,\end{aligned}$$

which implies that

$$\begin{aligned}\int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx &\geq \int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} (w_\epsilon - dm_\epsilon) \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx + o_\epsilon(1) \\ &\geq \int_{\mathbb{B}_R(0)} (w_\epsilon - dm_\epsilon) \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} r_\epsilon^n\,dx + o_\epsilon(1) \\ &\geq (1-d) \int_{\mathbb{B}_R(0)} \exp\{\tfrac{n}{n-1}\alpha_n \psi\}\,dx + o_\epsilon(1) + o_\epsilon(R)\end{aligned}$$

where $o_\epsilon(R) \to 0$ as $\epsilon \to 0$ for any fixed $R > 0$. Considering (3.7), let $\epsilon \to 0$, then $R \to \infty$,

$$\liminf_{\epsilon\to 0} \int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx \geq 1 - d. \tag{3.11}$$

By the same argument, we establish that

$$\liminf_{\epsilon\to 0} \int_\Omega |\nabla w_{\epsilon,d}|^n\,dx \geq d. \tag{3.12}$$

Since

$$\int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx + \int_\Omega |\nabla w_{\epsilon,d}|^n\,dx = 1,$$

by (3.12), we obtain

$$\limsup_{\epsilon\to 0} \int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx \leq 1 - d.$$

Combining the above inequality and (3.11), we deduce that

$$\lim_{\epsilon\to 0} \int_\Omega |\nabla(w_\epsilon - dm_\epsilon)^+|^n\,dx = 1 - d.$$

Then

$$\lim_{\epsilon\to 0} \int_\Omega |\nabla w_{\epsilon,d}|^n\,dx = d.$$

The following lemma is used in proving the existence of extremal functions of the improved Moser-Trudinger inequality with L^p norm. Since it provides the asymptotic behavior of w_ϵ, we include it here.

Lemma 3.8

$$\limsup_{\epsilon\to 0}\int_\Omega \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx \le |\Omega| + \lim_{R\to\infty}\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx. \tag{3.13}$$

Proof. For any $0<d<1$, from the expression of λ_ϵ in (3.2),

$$\begin{aligned}\int_\Omega \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx &= \int_{w_\epsilon<dm_\epsilon} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx + \int_{w_\epsilon\ge dm_\epsilon} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx\\ &\le \int_\Omega \exp\{\alpha_\epsilon w_{\epsilon,d}^{\frac{n}{n-1}}\}\,dx + \frac{\lambda_\epsilon}{(dm_\epsilon)^{\frac{n}{n-1}}}.\end{aligned}$$

Thanks to (3.9) and Lions' result, $\exp\{\alpha_\epsilon w_{\epsilon,d}^{\frac{n}{n-1}}\}$ in bounded in $L^s(\Omega)$ for some $s>1$. Since $w_{\epsilon,d}\to 0$ a.e. in Ω as $\epsilon\to 0$ in Lemma 3.4,

$$\int_\Omega \exp\{\alpha_\epsilon w_{\epsilon,d}^{\frac{n}{n-1}}\}\,dx \to \int_\Omega \exp\{0\}\,dx = \Omega, \quad \text{as } \epsilon\to 0.$$

Let $\epsilon\to 0$, then $d\to 1$ and

$$\limsup_{\epsilon\to 0}\int_\Omega \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx \le |\Omega| + \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{m_\epsilon^{\frac{n}{n-1}}}. \tag{3.14}$$

On the other hand, from the r_ϵ in (3.4),

$$\int_{\mathbb{B}_{Rr_\epsilon(x_\epsilon)}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx = \frac{\lambda_\epsilon}{\beta_\epsilon m_\epsilon^{\frac{n}{n-1}}}\left(\int_{\mathbb{B}_R(0)} \exp\{\frac{n\alpha_n}{n-1}\psi\}\,dx + o_\epsilon(R)\right), \tag{3.15}$$

where $o_\epsilon(R)\to 0$ as $\epsilon\to 0$ for any fixed $R>0$. Taking $\epsilon\to 0$, then $R\to\infty$ and by the fact that $\beta_\epsilon\to 1$,

$$\lim_{R\to\infty}\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon(x_\epsilon)}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx = \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{m_\epsilon^{\frac{n}{n-1}}}. \tag{3.16}$$

Together with (3.14), the lemma is completed.

By splitting Ω into three parts

$$\Omega = (\{w_\epsilon > dm_\epsilon\}\setminus \mathbb{B}_{Rr_\epsilon}(x_\epsilon)) \cup \{w_\epsilon \le dm_\epsilon\} \cup \mathbb{B}_{Rr_\epsilon}(x_\epsilon)$$

for some $0<d<1$, we get

Lemma 3.9 *$\forall\varphi\in C_0^\infty(\Omega)$,*

$$\lim_{\epsilon\to 0}\int_\Omega \varphi\beta_\epsilon\lambda_\epsilon^{-1} m_\epsilon w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx = \varphi(z).$$

Lemma 3.10 *If $\|f_\epsilon\|_1\le C$, and $v_\epsilon\in C^1(\bar\Omega)\cap H^{1,n}(\Omega)$ satisfies*

$$-\triangle_n v_\epsilon = f_\epsilon + \alpha|v_\epsilon|^{p-2}v_\epsilon \ \text{ in } \Omega, \tag{3.17}$$

where $0\le\alpha\le\bar\lambda(\Omega)$, then for $1<q<n$,

$$\|\nabla v_\epsilon\|_q \le C_1(q,n,\Omega,C).$$

The above two lemmas are similar to those in [11] and [24]. We omit their proofs here.

Lemma 3.11 *For any $1 < q < n$, $m_\epsilon^{\frac{1}{n-1}} w_\epsilon$ is bounded in $H_0^{1,q}(\Omega)$.*

Proof. By (3.1), we have

$$\begin{aligned} -\triangle_n(m_\epsilon^{\frac{1}{n-1}} w_\epsilon) &= m_\epsilon \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} + \gamma_\epsilon m_\epsilon \|w_\epsilon\|_p^{n-p} w_\epsilon^{p-1} \\ &= m_\epsilon \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} + \gamma_\epsilon \|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p^{n-p} (m_\epsilon^{\frac{1}{n-1}} w_\epsilon)^{p-1}. \end{aligned} \tag{3.18}$$

We claim that $\|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p$ is bounded. Otherwise, $\|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p \to \infty$ as $\epsilon \to 0$. We show it is impossible. Set

$$\widehat{w_\epsilon} = m_\epsilon^{\frac{1}{n-1}} w_\epsilon / \|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p,$$

then $\|\widehat{w_\epsilon}\|_p = 1$ and

$$-\triangle_n(\widehat{w_\epsilon}) = \frac{m_\epsilon \beta_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}}{\|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p^{n-1}} + \gamma_\epsilon \widehat{w_\epsilon}^{p-1}. \tag{3.19}$$

Thanks to Lemma 3.10, we conclude that $\|\nabla \widehat{w_\epsilon}\|_q \le C$ which is independent of ϵ for any $1 < q < n$. We may assume that there exists $\hat{w} \in H_0^{1,q}(\Omega)$ such that

$$\widehat{w_\epsilon} \rightharpoonup \hat{w} \text{ weakly in } H_0^{1,q}(\Omega),$$

$$\widehat{w_\epsilon} \to \hat{w} \text{ stronly in } L^p(\Omega),$$

as $\epsilon \to 0$. Multiplying (3.19) by $\varphi \in C_0^1(\Omega)$ and taking $\epsilon \to 0$, we have

$$\int_\Omega |\nabla \hat{w}|^{n-2} \nabla \hat{w} \nabla \varphi \, dx = \alpha \int_\Omega \hat{w}^{p-1} \varphi \, dx.$$

In above, Lemma 3.9 is applied. Since $0 \le \alpha < \bar{\lambda}(\Omega)$, we can easily deduce that $\hat{w} \equiv 0$. Nevertheless, $\|\hat{w}\|_p = 1$, which immediately implies that a contradiction exists. The claim is verified. Using the Lemma 3.10 again, we show that $m_\epsilon^{\frac{1}{n-1}} w_\epsilon$ is bounded in $H_0^{1,q}(\Omega)$ for any $1 < q < n$.

Lemma 3.12 *For any any $1 < q < n$, $m_\epsilon^{\frac{1}{n-1}} w_\epsilon \rightharpoonup G$ weakly in $H_0^{1,q}(\Omega)$ as $\epsilon \to 0$, where $G \in C^1(\Omega \backslash \{z\})$ is the Green function satisfying*

$$\begin{cases} -\triangle_n G = \delta_z + \alpha \|G\|_p^{n-p} G^{p-1} & \text{in } \Omega, \\ G = 0 & \text{on } \Omega. \end{cases} \tag{3.20}$$

Furthermore, $m_\epsilon^{\frac{1}{n-1}} w_\epsilon \to G$ in $C^1(\bar{\Omega}')$ for any domain $\Omega' \Subset (\bar{\Omega} \backslash \{z\})$.

Proof. Since $m_\epsilon^{\frac{1}{n-1}} w_\epsilon$ is shown to be bounded in $H_0^{1,q}(\Omega)$, we may assume that there exists $G(x)$ such that

$$m_\epsilon^{\frac{1}{n-1}} w_\epsilon \rightharpoonup G$$

weakly in $H_0^{1,q}(\Omega)$ as $\epsilon \to 0$. Testing (3.18) by $\varphi \in C_0^\infty(\Omega)$, with the help of Lemma 3.9, we obtain

$$\begin{aligned} \int_\Omega |\nabla(m_\epsilon^{\frac{1}{n-1}} w_\epsilon)|^{n-2} \nabla(m_\epsilon^{\frac{1}{n-1}} w_\epsilon) \nabla \varphi \, dx &= \int_\Omega \varphi \beta_\epsilon m_\epsilon \lambda_\epsilon^{-1} w_\epsilon^{\frac{1}{n-1}} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} \\ &\quad + \varphi \gamma_\epsilon \|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p^{n-p} (m_\epsilon^{\frac{1}{n-1}} w_\epsilon)^{p-1} \, dx \\ &\to \varphi(z) + \alpha \int_\Omega \|G\|_p^{n-p} G^{p-1} \varphi \, dx. \end{aligned}$$

Hence

$$\int_\Omega |\nabla G|^{n-2}\nabla G\nabla\varphi\, dx = \varphi(z) + \alpha\|G\|_p^{n-p}\int_\Omega G^{p-1}\varphi\, dx, \tag{3.21}$$

that is,

$$-\triangle_n G = \delta_z + \alpha\|G\|_p^{n-p}G^{p-1}.$$

For any fixed small δ, we choose a cut-off function $\xi(x) \in C_0^\infty(\Omega \setminus \mathbb{B}_\delta(z))$ such that $\xi(x) \equiv 1$ on $\Omega \setminus \mathbb{B}_{3\delta}(z)$. By Lemma 3.4, we get

$$\|\nabla(\xi w_\epsilon)\|_n \to 0$$

as $\epsilon \to 0$. Then $\exp\{(\xi w_\epsilon)^{\frac{n}{n-1}}\}$ is bounded in $L^s(\Omega \setminus \mathbb{B}_\delta(z))$ for any $s > 1$. Furthermore, $\exp\{w_\epsilon^{\frac{n}{n-1}}\}$ is bounded in $L^s(\Omega \setminus \mathbb{B}_{3\delta}(z))$. Since $\|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_p^{n-p}(m_\epsilon^{\frac{1}{n-1}} w_\epsilon)^{p-1}$ is bounded $L^{\frac{p}{p-1}}(\Omega)$, note that $\|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_n$ is bounded, we obtain

$$\|m_\epsilon^{\frac{1}{n-1}} w_\epsilon\|_\infty < C$$

in $\bar\Omega \setminus \mathbb{B}_{4\delta}(z)$ by applying the classical elliptic estimate. Furthermore,

$$m_\epsilon^{\frac{1}{n-1}} w_\epsilon \to G$$

in $C^1(\bar\Omega \setminus \mathbb{B}_{5\delta}(z))$ as $\epsilon \to 0$.

So far, we have characterized the asymptotic behavior of w_ϵ in the case that the concentration point lies in the interior of Ω. Next we investigate the situation when z lies on $\partial\Omega$. The main idea is almost the same as the case 1. We only show the differences below.

Case 2, z lies on $\partial\Omega$.

Lemma 3.13 *Let $d_\epsilon = dist(x_\epsilon, \partial\Omega)$ and be r_ϵ in (3.4). Then $r_\epsilon d_\epsilon^{-1} \to 0$.*

Proof. We prove it by contradiction. Suppose there exists some R such that $d_\epsilon \le Rr_\epsilon$, i.e. $1/R < dr_\epsilon^{-1}$. As we know, there exists a unique $y_\epsilon \in \partial\Omega$ such that $d_\epsilon = |x_\epsilon - y_\epsilon|$. Define

$$\bar v_\epsilon(x) = m_\epsilon^{-1} w_\epsilon(y_\epsilon + r_\epsilon x).$$

By a reflection argument and elliptic estimates, we obtain $\bar v_\epsilon(x) \to 1$ in $C^1(\overline{\mathbb{B}_R^+})$, which contradicts the fact that $\bar v_\epsilon(0) = 0$ as $\epsilon \to 0$. Therefore, the conclusion holds.

Let $\psi_\epsilon(x)$ be in (3.4) and ψ be in (3.7). The above lemma also justifies that $\psi_\epsilon(x) \to \psi$ in $C^1_{loc}(\mathbb{R}^n)$. Arguing as the case 1, $m_\epsilon^{\frac{1}{n-1}} w_\epsilon \rightharpoonup \bar G$ weakly in $H^{1,q}(\Omega)$ and in $C^1(\Omega)$, where $\bar G$ satisfies

$$\begin{cases} -\triangle_n \bar G = \alpha\|\bar G\|_p^{n-p}\bar G^{p-1} & \text{in } \Omega, \\ \bar G = 0 & \text{on } \Omega. \end{cases} \tag{3.22}$$

Since $0 \le \alpha < \bar\lambda(\Omega)$, then $\bar G \equiv 0$. Hence

$$m_\epsilon^{\frac{1}{n-1}} w_\epsilon \rightharpoonup 0 \text{ weakly in } H^{1,q}(\Omega),$$

$$m_\epsilon^{\frac{1}{n-1}} w_\epsilon \to 0 \text{ in } C^1(\bar\Omega \setminus z)$$

as $\epsilon \to 0$.

With those two cases considered, we are able to finish the proof of Theorem 1.

Proof. [Proof of conclusion (i) in Theorem 1]

If m_ϵ is bounded, as we discussed before, the conclusion (i) of Theorem 1 holds directly. If $m_\epsilon \to \infty$, we have $\|w_\epsilon\|_n \to 0$ from Lemma 3.4. Then

$$\begin{aligned} I^{\alpha}_{\alpha_n-\epsilon}(w_\epsilon) &= \int_\Omega \exp\{(\alpha_n-\epsilon)|w_\epsilon|^{\frac{n}{n-1}}((1+\alpha\|w_\epsilon\|_p^n)^{\frac{1}{n-1}}-1)\}\exp\{(\alpha_n-\epsilon)|w_\epsilon|^{\frac{n}{n-1}}\}\,dx \\ &\le \exp\{\alpha_n m_\epsilon^{\frac{n}{n-1}}((1+\alpha\|w_\epsilon\|_p^n)^{\frac{1}{n-1}}-1)\int_\Omega \exp\{(\alpha_n-\epsilon)|w_\epsilon|^{\frac{n}{n-1}}\}\,dx \\ &\le \exp\{\tfrac{\alpha_n\alpha}{n-1}\|m_\epsilon^{\frac{1}{n-1}}w_\epsilon\|_p^n + m_\epsilon^{\frac{-n}{n-1}}O(\|m_\epsilon^{\frac{1}{n-1}}w_\epsilon\|_p^{2n})\}\int_\Omega \exp\{(\alpha_n-\epsilon)|w_\epsilon|^{\frac{n}{n-1}}\}\,dx. \end{aligned}$$

If $z\in\Omega$, it is known that

$$\|m_\epsilon^{\frac{1}{n-1}}w_\epsilon\|_p^n \to \|G\|_p^n.$$

With the aid of the classical Moser-Trudinger inequality and Lemma 3.2, we derive the conclusion (i) in Theorem 1. If $z\in\partial\Omega$, as in the case 2, $\|m_\epsilon^{\frac{1}{n-1}}w_\epsilon\|_p^n \to 0$. The same results follow.

4 Existence of Moser-Trudinger Functions

In this section, we show that the existence of the extremal functions of the improved Moser-Trudinger inequality involving L^p norm in n dimensions. We divide the proof Theorem 2 into two steps. In the first step, we derive a upper bound for $I^{\alpha}_{\alpha_n}$. Two cases have to be considered as Section 3, that is, z lies in the interior of Ω and z lies on $\partial\Omega$. Recall that

$$m_\epsilon = w_\epsilon(x_\epsilon) = \max_\Omega w_\epsilon(x).$$

Step 1: (The upper bound for $I^{\alpha}_{\alpha_n}$) Under the assumption that $m_\epsilon\to\infty$ and $x_\epsilon\to z\in\Omega$, the following holds

$$\sup_{w\in H_0^{1,n}(\Omega),\ \|\nabla w\|_n=1}\int_\Omega \exp\{\alpha_\epsilon w^{\frac{n}{n-1}}\}\,dx \le |\Omega| + \frac{\omega_{n-1}}{n}\exp\{\alpha_n A_z + 1 + \frac{1}{2} + \cdots \frac{1}{n-1}\}, \tag{4.1}$$

where A_z is defined in (4.2).

Case 1, z lies in the interior of Ω.

Similar to [11], [12] and [24], the theorem of Carleson and Chang [3] plays an important role.

Theorem B: (Carleson and Chang)

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$. Assume that w_ϵ is a sequence of function in $H_0^{1,n}(\mathbb{B})$ with $\|\nabla w_\epsilon\|_n = 1$. If $|\nabla w_\epsilon| \rightharpoonup \delta_0$ weakly in sense of measure as $\epsilon\to 0$, then

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}} \exp\{\alpha_n |w_\epsilon|^{\frac{n}{n-1}}\}\,dx \le |\mathbb{B}|(1+\exp\{1+\frac{1}{2}+\cdots\frac{1}{n-1}\}).$$

As in [24], we have the following representation of $G(x)$ in Lemma 3.12. $G(x)$ can be represented as

$$G(x) = -\frac{n}{\alpha_n}\log|x-z| + A_z + B(x), \tag{4.2}$$

where A_z is a constant, $B(x)$ is continuous at z, $B(z)=0$ and $B(x)\in C^1(\bar{\Omega}\backslash z)$.

For simplicity we denote $\mathbb{B}_\delta(z) = \{x\in\mathbb{R}^n : |x-z|\le\delta\}$ by $\mathbb{B}_\delta$, and $\partial\mathbb{B}_\delta(z)$ by $\partial\mathbb{B}_\delta$. With the aid of Lemma 3.12, we have

$$\begin{aligned} \int_{\Omega\backslash\mathbb{B}_\delta}|\nabla w_\epsilon|^n\,dx &= m_\epsilon^{\frac{-n}{n-1}}(\int_{\Omega\backslash\mathbb{B}_\delta}|\nabla G|^n\,dx + o_\epsilon(1)) \\ &= m_\epsilon^{\frac{-n}{n-1}}(\alpha\int_{\Omega\backslash\mathbb{B}_\delta}\|G\|_p^{n-p}G^p\,dx + \int_{\partial\mathbb{B}_\delta}G|\nabla G|^{n-2}\tfrac{\partial G}{\partial n}\,ds + o_\epsilon(1)) \\ &= m_\epsilon^{\frac{-n}{n-1}}(\tfrac{-n}{\alpha_n}\log\delta + \alpha\|G\|_p^n + A_z + B(\xi) + o_\delta(1) + o_\epsilon(1)), \end{aligned}$$

where (4.2) is used and $\xi \in \partial\mathbb{B}_\delta$. Because of the continuity of $B(\xi)$, we obtain that

$$\int_{\Omega\setminus\mathbb{B}_\delta} |\nabla w_\epsilon|^n\,dx = m_\epsilon^{\frac{-n}{n-1}}\left(\frac{-n}{\alpha_n}\log\delta + \alpha\|G\|_p^n + A_z + o_\delta(1) + o_\epsilon(1)\right). \tag{4.3}$$

Let $e_\epsilon = \sup_{\partial\mathbb{B}_\delta} w_\epsilon$ and $\bar{w}_\epsilon = (w_\epsilon - e_\epsilon)^+$, then $\bar{w}_\epsilon \in H_0^{1,n}(\mathbb{B}_\delta)$. Since

$$\int_{\mathbb{B}_\delta} |\nabla w_\epsilon|^n\,dx = 1 - \int_{\Omega\setminus\mathbb{B}_\delta} |\nabla w_\epsilon|^n\,dx$$

and

$$\int_{\mathbb{B}_\delta} |\nabla \bar{w}_\epsilon|^n\,dx \le \int_{\mathbb{B}_\delta} |\nabla w_\epsilon|^n\,dx,$$

based on (4.3), we get that

$$\int_{\mathbb{B}_\delta} |\nabla \bar{w}_\epsilon|^n\,dx \le \tau_\epsilon := 1 - m_\epsilon^{\frac{-n}{n-1}}\left(\frac{-n}{\alpha_n}\log\delta + \alpha\|G\|_p^n + A_z + o_\delta(1) + o_\epsilon(1)\right). \tag{4.4}$$

By the Theorem B of Carleson-Chang,

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}_\delta} \exp\{\alpha_n |w_\epsilon/\tau_\epsilon^{\frac{1}{n}}|^{\frac{n}{n-1}}\}\,dx \le \frac{\omega_{n-1}\delta^n}{n}\left(1 + \exp\{1 + \frac{1}{2} + \cdots \frac{1}{n-1}\}\right). \tag{4.5}$$

Next we concentrate on the behavior of w_ϵ on $\mathbb{B}_{Rr_\epsilon}(x_\epsilon)$. Recall α_ϵ in (3.2). Due to Lemma 3.12 and the representation of $G(x)$ in (4.2), we have

$$\begin{aligned}
\alpha_\epsilon |w_\epsilon|^{\frac{n}{n-1}} &\le \alpha_n (1 + \alpha\|w_\epsilon\|_p^n)^{\frac{1}{n-1}} (\bar{w}_\epsilon + e_\epsilon)^{\frac{n}{n-1}} \\
&\le \alpha_n \bar{w}_\epsilon^{\frac{n}{n-1}} + \tfrac{n\alpha_n}{n-1}\bar{w}_\epsilon^{\frac{1}{n-1}} e_\epsilon + \tfrac{\alpha\alpha_n}{n-1}\|G\|_p^n + o_\epsilon(1) \\
&\le \alpha_n \bar{w}_\epsilon^{\frac{n}{n-1}} + \tfrac{\alpha\alpha_n}{n-1}\|G\|_p^n - \tfrac{n^2}{n-1}\log\delta + \tfrac{n\alpha_n}{n-1}A_z + o_\epsilon(1) + o_\delta(1) \\
&\le \frac{\alpha_n \bar{w}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{1}{n-1}}} + \alpha_n A_z - \log\delta^n + o_\epsilon(1) + o_\delta(1).
\end{aligned}$$

Integrating the above estimates on $\mathbb{B}_{Rr_\epsilon}(x_\epsilon)$, we establish that

$$\begin{aligned}
\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx &\le \delta^{-n}\exp\{\alpha_n A_z + o_\epsilon(1)\}\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} \exp\{\tfrac{\alpha_\epsilon \bar{w}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{1}{n-1}}}\}\,dx \\
&\le \delta^{-n}\exp\{\alpha_n A_z + o_\epsilon(1)\}\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} (\exp\{\tfrac{\alpha_\epsilon \bar{w}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{1}{n-1}}}\} - 1)\,dx + o_\epsilon(1) \\
&\le \delta^{-n}\exp\{\alpha_n A_z + o_\epsilon(1)\}\int_{\mathbb{B}_\delta} (\exp\{\tfrac{\alpha_\epsilon \bar{w}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{1}{n-1}}}\} - 1)\,dx.
\end{aligned}$$

Following from (4.5),

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx \le \frac{\omega_{n-1}}{n}\exp\{\alpha_n A_z + 1 + \frac{1}{2} + \cdots \frac{1}{n-1}\}. \tag{4.6}$$

Thanks to Lemma 3.8, we deduce that

$$\limsup_{\epsilon\to 0}\int_{\Omega} \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\}\,dx \le |\Omega| + \frac{\omega_{n-1}}{n}\exp\{\alpha_n A_z + 1 + \frac{1}{2} + \cdots \frac{1}{n-1}\}. \tag{4.7}$$

So we complete the step 1 for the case that the limit point z lies in Ω.

Case 2, z lies on $\partial\Omega$.

We argue as the case 1. Since $m_\epsilon^{\frac{1}{n-1}} w_\epsilon \rightharpoonup \bar{G} \equiv 0$ weakly in $H^{1,q}(\Omega)$ for any $1 < q < n$ and in $C^1(\bar{\Omega} \setminus z)$, we have

$$\int_{\mathbb{B}_\delta} |\nabla \bar{w}_\epsilon|^n \, dx \le \tau_\epsilon = 1 - o_\epsilon(1) m_\epsilon^{\frac{-n}{n-1}}.$$

On the domain $\mathbb{B}_{Rr_\epsilon}(x_\epsilon)$, we derive

$$\alpha_\epsilon |w_\epsilon|^{\frac{n}{n-1}} \le \alpha_n |\bar{w}_\epsilon / \tau_\epsilon^{\frac{1}{n}}|^{\frac{n}{n-1}} + o_\epsilon(1).$$

Combining the above estimates and Lemma 3.8, we obtain

$$\limsup_{\epsilon \to 0} \int_\Omega \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} \, dx \le |\Omega| + O(\delta^n) \exp\{1 + \frac{1}{2} + \cdots \frac{1}{n-1}\}.$$

Taking $\delta \to 0$,

$$\limsup_{\epsilon \to 0} \int_\Omega \exp\{\alpha_\epsilon w_\epsilon^{\frac{n}{n-1}}\} \, dx \le |\Omega|,$$

which obviously is a contradiction. Therefore, it indicates that z can not lie on $\partial\Omega$.

In conclusion, the step 1 is shown.

In the second step, we construct an explicit test function. This test function provides a lower bound for improved Moser-Trudinger inequality involving L^p norm, which has the exactly same value of the upper bound. By exploring this contradiction, we arrive at the fact that the blow-up sequence do not exist, that is, m_ϵ is bounded in Ω. Hence, Theorem 2 follows.

Step 2: (The Lower bound for $I_{\alpha_n}^\alpha$) There exists $\varphi_\epsilon \in \mathcal{H}$ such that

$$\int_\Omega \exp\{\alpha_n |\varphi_\epsilon|^{\frac{n}{n-1}} (1 + \alpha \|\varphi_\epsilon\|_p^n)^{\frac{1}{n-1}}\} \, dx > |\Omega| + \frac{\omega_{n-1}}{n} \exp\{\alpha_n A_z + 1 + \frac{1}{2} + \cdots \frac{1}{n-1}\}.$$

Let $r(x) = |x - z|$, where z is the concentration point. Set $\tilde{G} = G + \frac{n \log r(x)}{\alpha_n} - A_z$. It is easy to see that $\tilde{G} = O(r(x))$. Define

$$\varphi_\epsilon = \begin{cases} \frac{c + c^{\frac{-1}{n-1}} (-\frac{n-1}{\alpha_n} \log(1 + c_n (\frac{r(x)}{\epsilon})^{\frac{n}{n-1}}) + B)}{(1 + \alpha c^{\frac{-n}{n-1}} \|G\|_p^n)^{\frac{1}{n}}} & \text{for } \; r(x) \le R\epsilon, \\ \frac{G - \eta \tilde{G}}{(c^{\frac{n}{n-1}} + \alpha \|G\|_p^n)^{\frac{1}{n}}} & \text{for } \; R\epsilon < r(x) \le 2R\epsilon, \\ \frac{G}{(c^{\frac{n}{n-1}} + \alpha \|G\|_p^n)^{\frac{1}{n}}} & \text{for } \; 2R\epsilon < r(x), \end{cases} \tag{4.8}$$

where $c_n = (\frac{\omega_{n-1}}{n})^{\frac{1}{n-1}}$, $\eta \in C_0^\infty(\mathbb{B}_{2R\epsilon}(z))$ is a cut-off function with $\eta = 1$ on $\mathbb{B}_{r\epsilon}(z)$ and $\|\nabla \eta\|_\infty = O(\frac{1}{R\epsilon})$, B is a constant to be determined, and R, c depending on ϵ will also be determined such that $R\epsilon \to 0$ and $R \to \infty$. Since $\varphi_\epsilon \in H_0^{1,n}(\Omega)$, we have

$$c + c^{\frac{-1}{n-1}} (-\frac{n-1}{\alpha_n} \log(1 + c_n (\frac{r(x)}{\epsilon})^{\frac{n}{n-1}}) + B) = \frac{\frac{-n}{\alpha_n} \log R\epsilon + A_z}{c^{\frac{1}{n-1}}},$$

which implies that

$$c^{\frac{n}{n-1}} = \frac{-n}{\alpha_n} \log \epsilon + \frac{n-1}{\alpha_n} \log c_n - B + A_z + O(R^{\frac{-n}{n-1}}). \tag{4.9}$$

Next we make sure that $\int_\Omega |\nabla\varphi_\epsilon|^n\,dx = 1$.

$$\begin{aligned}
\int_{r(x)\le R\epsilon} |\nabla\varphi_\epsilon|^n\,dx &= \tfrac{n-1}{\alpha_n(c^{\frac{n}{n-1}}+\alpha\|G\|_p^n)}\int_0^{c_nR^{\frac{n}{n-1}}}\tfrac{z^{n-1}}{(1+z)^n}\,dz\\
&= \tfrac{n-1}{\alpha_n(c^{\frac{n}{n-1}}+\alpha\|G\|_p^n)}\int_0^{c_nR^{\frac{n}{n-1}}}\tfrac{((1+z)-1)^{n-1}}{(1+z)^n}\,dz\\
&= \tfrac{n-1}{\alpha_n(c^{\frac{n}{n-1}}+\alpha\|G\|_p^n)}\left(\textstyle\sum_{k=0}^{n-2}\tfrac{C_{n-1}^k(-1)^{n-1-k}}{n-k-1} + \log(1+c_nR^{\frac{n}{n-1}})\right.\\
&\quad \left.+O(R^{\frac{-n}{n-1}})\right)\\
&= \tfrac{n-1}{\alpha_n(c^{\frac{n}{n-1}}+\alpha\|G\|_p^n)}\left(-(1+\tfrac{1}{2}+\cdots+\tfrac{1}{n-1}) + \log(1+c_nR^{\frac{n}{n-1}})\right.\\
&\quad \left.+O(R^{\frac{-n}{n-1}})\right),
\end{aligned}$$

where we have used the fact that

$$-\sum_{k=0}^{n-2}\frac{C_{n-1}^k(-1)^{n-1-k}}{n-k-1} = 1+\frac{1}{2}+\cdots+\frac{1}{n-1}.$$

Taking into account the expression of (4.8), then

$$\begin{aligned}
\int_\Omega |\nabla\varphi_\epsilon|^n\,dx &= \frac{n-1}{\alpha_n(c^{\frac{n}{n-1}}+\alpha\|G\|_p^n)}\Big(\frac{-n}{n-1}\log\epsilon+\log c_n+\frac{\alpha_n}{n-1}A_z\\
&\quad +\frac{\alpha\alpha_n}{n-1}\|G\|_p^n-(1+\frac{1}{2}+\cdots+\frac{1}{n-1})\\
&\quad +O(R^{\frac{-n}{n-1}})+O(R\epsilon\log(R\epsilon))\Big).
\end{aligned} \tag{4.10}$$

Since $\int_\Omega |\nabla\varphi_\epsilon|^n\,dx = 1$, by (4.10),

$$\begin{aligned}
c^{\frac{n}{n-1}} &= \frac{-n}{\alpha_n}\log\epsilon+\frac{n-1}{\alpha_n}\log c_n+A_z-(n-1)(1+\frac{1}{2}+\cdots+\frac{1}{n-1})\\
&\quad +O(R^{\frac{-n}{n-1}})+O(R\epsilon\log(R\epsilon)).
\end{aligned} \tag{4.11}$$

From (4.9),

$$B=(n-1)(1+\frac{1}{2}+\cdots+\frac{1}{n-1})+O(R^{\frac{-n}{n-1}})+O(R\epsilon\log(R\epsilon)). \tag{4.12}$$

Set $R=-\log\epsilon$, which satisfies $R\epsilon\to 0$ as $\epsilon\to 0$. Since

$$\|\varphi_\epsilon\|_p^n = \frac{\|G\|_p^n+O(c^{\frac{n^2}{n-1}}R^{\frac{n^2}{p}}\epsilon^{\frac{n^2}{p}})+O((R\epsilon)^{\frac{n^2}{p}}(-\log(R\epsilon))^n)}{c^{\frac{n}{n-1}}+\alpha\|G\|_p^n},$$

using the inequality

$$(1+t)^{\frac{-1}{n-1}}\ge 1-\frac{t}{n-1}$$

for t small, (4.9) and (4.12),

$$\begin{aligned}\alpha_n|\varphi_\epsilon|^{\frac{n}{n-1}}(1+\alpha\|\varphi_\epsilon\|_p^n)^{\frac{1}{n-1}} &\geq \alpha_n c^{\frac{n}{n-1}} - n\log(1+c_n(\frac{r(x)}{\epsilon})^{\frac{n}{n-1}}) + \frac{n\alpha_n}{n-1}B \\ &\quad -\frac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + O(c^{\frac{-2n}{n-1}}) + O(c^{\frac{n^2}{n-1}}R^{\frac{n^2}{p}}\epsilon^{\frac{n^2}{p}}) \\ &\quad +O((R\epsilon)^{\frac{n^2}{p}}(-\log(R\epsilon))^n) \\ &\geq -n\log\epsilon + (n-1)\log c_n + D - n\log(1+c_n(\frac{r(x)}{\epsilon})^{\frac{n}{n-1}}) \\ &\quad -\frac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L \end{aligned} \tag{4.13}$$

on $\mathbb{B}_{R\epsilon}(z)$, where

$$D := \alpha_n A_z + (1 + \frac{1}{2} + \cdots + \frac{1}{n-1})$$

and

$$L := O(c^{\frac{-2n}{n-1}}) + O(c^{\frac{n^2}{n-1}}R^{\frac{n^2}{p}}\epsilon^{\frac{n^2}{p}}) + O(R\epsilon(\log(R\epsilon))) + O(R^{\frac{-n}{n-1}}).$$

With the above estimates, we have

$$\begin{aligned}&\int_{\mathbb{B}_{R\epsilon}} \exp\{\alpha_n|\varphi_\epsilon|^{\frac{n}{n-1}}(1+\alpha\|\varphi_\epsilon\|_p^n)^{\frac{1}{n-1}}\}\,dx \\ &\geq \exp\{-n\log\epsilon + (n-1)\log c_n + D - \tfrac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L\} \\ &\quad \times \int_{\mathbb{B}_{R\epsilon}} \exp\{-n\log(1+c_n(\tfrac{r(x)}{\epsilon})^{\frac{n}{n-1}})\}\,dx \\ &\geq c_n^{n-1}\exp\{D - \tfrac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L\}\int_{\mathbb{B}_{R\epsilon}} \tfrac{\epsilon^{-n}}{(1+c_n(\frac{r(x)}{\epsilon})^{\frac{n}{n-1}})^n}\,dx \\ &\geq \tfrac{(n-1)\omega_{n-1}}{n}\exp\{D - \tfrac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L\}\int_0^{c_nR^{\frac{n}{n-1}}} \tfrac{z^{n-2}}{(1+z)^n}\,dz \\ &\geq \tfrac{(n-1)\omega_{n-1}}{n}\exp\{D - \tfrac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L\}\int_0^{c_nR^{\frac{n}{n-1}}} \tfrac{((1+z)-1)^{n-2}}{(1+z)^n}\,dz \\ &\geq \tfrac{(n-1)\omega_{n-1}}{n}\exp\{D - \tfrac{\alpha_n\alpha^2\|G\|_p^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L\}(\tfrac{1}{n-1} + O(R^{\frac{-n}{n-1}})) \\ &\geq \tfrac{\omega_{n-1}}{n}\exp\{D\} - \exp\{D\}\tfrac{\omega_{n-1}\alpha_n\alpha^2\|G\|_p^{2n}}{n(n-1)c^{\frac{n}{n-1}}} + L,\end{aligned}$$

where we have applied the fact that

$$\sum_{k=0}^{n-2}\frac{C_{n-2}^k(-1)^{n-k-2}}{n-k-1} = \frac{1}{n-1}.$$

On the other hand

$$\begin{aligned}\int_{\Omega\setminus\mathbb{B}_{R\epsilon}(z)} \exp\{\alpha_n|\varphi_\epsilon|^{\frac{n}{n-1}}(1+\alpha\|\varphi_\epsilon\|_p^n)^{\frac{1}{n-1}}\}\,dx &\geq \int_{\Omega\setminus\mathbb{B}_{2R\epsilon}}(1+\alpha_n|\varphi_\epsilon|^{\frac{n}{n-1}})\,dx \\ &\geq |\Omega| + \frac{\alpha_n\|G\|_{\frac{n}{n-1}}^{\frac{n}{n-1}}}{c^{\frac{n}{(n-1)^2}}} + O(R\epsilon) + O(c^{\frac{-2n}{(n-1)^2}}).\end{aligned}$$

Together with the above integral estimates on $\mathbb{B}_{R\epsilon}$, $(\Omega\setminus\mathbb{B}_{R\epsilon})$, and the fact that

$$L \to 0 \text{ and } O(c^{\frac{-2n}{(n-1)^2}}) \to 0,$$

we establish that

$$\int_{\Omega} \exp\{\alpha_n |\varphi_\epsilon|^{\frac{n}{n-1}} (1 + \alpha \|\varphi_\epsilon\|_p^n)^{\frac{1}{n-1}}\} \, dx > |\Omega| + \frac{\omega_{n-1}}{n} \exp\{\alpha_n A_z + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\} \tag{4.14}$$

for any $0 < \alpha < \bar{\lambda}(\Omega)$ and sufficient small ϵ in the case of $n \geq 3$, and for small enough α and sufficient small ϵ in the case of $n = 2$.

Proof. [Proof of Theorem 1.2] Combining the step 1 and step 2, we derive the existence of the extremal functions for the improved Moser-Trundiger inequality with L^p norm in n dimensions.

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