Quantitative uniqueness of elliptic equations

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ABSTRACT. Based on a variant of frequency function, we improve the vanishing order of solutions for Schrödinger equations which describes quantitative behavior of strong uniqueness continuation property. For the first time, we investigate the quantitative uniqueness of higher order elliptic equations and show the vanishing order of solutions. Furthermore, strong unique continuation is established for higher order elliptic equations using this variant of frequency function.

1. Introduction

We say the vanishing order of solution at x_0 is l, if l is the largest integer such that $D^\alpha u(x_0) = 0$ for all $|\alpha| \leq l$. It describes quantitative behavior of strong unique continuation property. It is well known that all zeros of nontrivial solutions of second order linear equations on smooth compact Riemannian manifolds are of finite order [**Ar**]. In the papers [**DF**] and [**DF1**], Donnelly and Fefferman showed that if u is an eigenfunction on a compact smooth Riemannian manifold $\mathcal{M}$, that is,

$$-\triangle_{\mathcal{M}} u = \lambda u, \qquad \text{in } \mathcal{M}$$

for some $\lambda > 0$, then the maximal vanishing order of u on $\mathcal{M}$ is less than $C\sqrt{\lambda}$, here C only depends on the manifold $\mathcal{M}$. Kukavica in [**Ku**] considered the vanishing order of solutions of Schrödinger equation

$$-\triangle_{\mathcal{M}} u = V(x)u, \tag{1.1}$$

where $V(x) \in L^\infty(\mathcal{M})$. He established that the vanishing order of solution in (1.1) is everywhere less than

$$C(1 + (\sup_\Omega V_+)^{\frac{1}{2}} + osc(V)^2), \tag{1.2}$$

where $V_+(x) = \max\{V(x),\ 0\}$, $osc(V) = \sup V - \inf V$ and C only depends on the underlying domain $\mathcal{M}$. If $V \in C^1(\mathcal{M})$, Kukavica was able to show that the upper bound of vanishing order is less than $C(1 + \|V\|_{C^1})$, where $\|V\|_{C^1} = \|V\|_{L^\infty} + \|\nabla V\|_{L^\infty}$. Based on the Donnelly and Fefferman's work in [**DF**], Kukavica conjectured that the rate of of vanishing order of u is less than $C(1 + \|V\|^{\frac{1}{2}}_{L^\infty})$ for the cases of $V \in L^\infty$ and $V \in C^1$. For the upper bound in (1.2), it agrees with Donnelly and Fefferman's results in the eigenvalue

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case $V(x)=\lambda$. Recently Kenig [**K**] considered a similar problem which is motivated by his work with Bourgain in [**BK**] on Anderson localization for the Bernoulli model. Kenig investigated the following normalized model. Let

$$\triangle u(x)=V(x)u(x) \quad \text{in } \mathbb{B}_{10}, \quad \text{with} \quad \|V\|_{L^\infty}\leq M \quad \text{and} \quad \|u\|_{L^\infty}\leq C_0, \tag{1.3}$$

where $\mathbb{B}_{10}$ is a ball centered at origin with radius 10 in $\mathbb{R}^n$. Assume that $\sup_{|x|\leq 1}|u(x)|\geq 1$ and $M>1$. Kenig established that

$$\|u\|_{L^\infty(\mathbb{B}_r)}\geq a_1 r^{a_2\beta(M)} \qquad \text{as } r\to 0, \tag{1.4}$$

where a_1, a_2 depend only on n, C_0 and $\beta(M)$ depends on M. By exploiting the Carleman estimates, Kenig proved the $\beta(M)=M^{\frac{2}{3}}$. He also pointed out that the exponent $\frac{2}{3}$ of M is sharp for complex valued V based on Meshkov's example in [**M**]. On the basis of Donnelly and Fefferman's work, Kenig asked if $\beta(M)=M^{\frac{1}{2}}$ can be achieved for real u, V. Very recently, Bakri in [**B**] considered (1.1) in the case of $V(x)\in C^1$. He obtained that the vanishing order of solutions in (1.1) is less than $C(1+\sqrt{\|V\|_{C^1}})$. His proof is an extension of the Carleman estimates in [**DF**]. It is worthwhile to mention that the vanishing order of solutions is closely related to the study of eigenfunctions on manifolds. We refer to the survey [**Z**] for detailed account.

We are especially interested in the model (1.3). Our first goal in this paper is to address the above problems. Relied on a variant of frequency function, we are able to verify that $\beta(M)=M^{\frac{1}{2}}$ is indeed true for the case of $V(x)\in W^{1,\infty}$ in (1.4). In particular, our result also confirms that the vanishing order of solutions in (1.1) is less than $C(1+\|V\|^{1/2}_{W^{1,\infty}})$ if $V\in W^{1,\infty}$.

Theorem 1. *Assume that $V(x)\in W^{1,\infty}(\mathbb{B}_{10})$. Under the assumptions in (1.3) with $\|V\|_{W^{1,\infty}}\leq M$, the maximal vanishing order of u in (1.3) is less than $C\sqrt{M}$, where C depends on n and C_0.*

Generally speaking, the Carleman estimates and frequency function are two principal ways to obtain quantitative uniqueness and strong unique continuation results for solutions of partial differential equations. Carleman estimates were introduced by Carleman, when he studied the strong unique continuation property. Carleman estimates are weighted integral inequalities. See e.g. [**H**], [**JK**], [**K**], [**KRS**], [**KT**], [**S**], [**W**], to just mention a few. In order to obtain the quantitative uniqueness results for solutions, one uses the Carleman estimates with a special choice of weight functions to obtain a type of Hadamard's three-ball theorem, then doubling estimates follow. The vanishing order will come from the doubling estimates. The frequency function was first observed by Almgren [**A**] for harmonic functions. The frequency function controls the local growth rate of u and is a local measure of its "degree" as a polynomial like function in $\mathbb{B}_r$. See e.g. [**GL**], [**GL1**], [**Lin**], [**HL**], [**Ku**], [**Ku1**], etc. Garofalo and Lin in [**GL**], [**GL1**] showed its powerful applications in strong unique continuation problem. The frequency function Garofalo and Lin investigated for equation (1.3) is given by

$$N(r)=\frac{rD(r)}{H(r)} \tag{1.5}$$

where $H(r)=\int_{\partial\mathbb{B}_r}u^2\,d\sigma$ and $D(r)=\int_{\mathbb{B}_r}|\nabla u|^2+Vu^2\,dx$. After one proves certain monotonicity of $N(r)$, the doubling estimates will follow by a standard argument. In [**GL**], it was shown that $e^{Cr}N(r)$ was monotone nondecreasing. However, C depends on the norm of V. It can not give the optimal bound for the vanishing order of solutions. Kukavica considered almost the same frequency function in [**Ku**]. He was able to move the norm of V away from the exponential, but it only gave the aforementioned bound $C(1+(\sup_\Omega V_+)^{\frac{1}{2}}+osc(V)^2)$ due to the limitations of the method. Some of the limitations come from the fact that one can not explore $H'(r)$ more because of its integration on the boundary of balls. Instead we consider a variant of (1.5). See our variant of frequency function for Schrödinger equations in section 2 and the frequency functions for high order elliptic equations in section 3 for the details. First, we establish a monotonicity property of this new variant of frequency function. Second, based on the monotonicity results, it leads to a L^2-version of Hadamard's three-ball theorem, which further implies a L^∞-version of Hadamard's three-ball theorem by elliptic estimates. At last, by a propagation of smallness argument, we derive the vanishing order of solutions.

Higher order elliptic equations are also important models in the study of partial differential equations. A nature question is to study the quantitative uniqueness of higher order elliptic equations. Our second goal is to investigate the vanishing order for solutions of higher order elliptic equations. We consider this normalized model:

$$(-\triangle)^m u(x) = \overline{V}(x)u(x) \quad \text{in } \mathbb{B}_{10} \quad \text{with} \quad \|\overline{V}\|_{L^\infty} \le M \quad \text{and} \quad \|u\|_{L^\infty} \le C_0. \tag{1.6}$$

We also assume that $\sup_{|x|\le 1}|u(x)| \ge 1$ and $M > 1$. To the best of our knowledge, the explicit vanishing order as Theorem 1 seems to be unknown for higher order elliptic equations. By exploiting this variant of frequency function, we are able to obtain the following theorem.

Theorem 2. *Assume that $\overline{V}(x) \in L^\infty(\mathbb{B}_{10})$ and $n \ge 4m$. Under the assumption in (1.6), the maximal order of vanishing of u in (1.6) is less than CM, where C depends on n, m and C_0.*

Unlike the Laplacian operator in (1.3), new difficulties arise since some kind of "symmetry" is lost for higher order elliptic equations. Our idea is to break the higher order elliptic equations into a system of semilinear equations. However, it still does not give the most desirable result as Theorem 1. We also develop a L^∞-version of Hadamard's three-ball theorem by exploring $W^{2m,p}$ estimates for higher order elliptic equations (see section 3 for the details). Compared with the frequency function argument, it seems to more difficult to obtain the vanishing order of solutions for higher order elliptic equations by Carleman estimates.

The quantitative uniqueness has applications in mathematical physics. For instance, the vanishing order of solutions plays an important role in [**BK**] on Anderson localization for the Bernoulli model. Suppose that u is a solution of

$$-\triangle u = Vu \quad \text{in } \mathbb{R}^n, \tag{1.7}$$

where $\|V\|_{L^\infty} \le 1$, $\|u\|_{L^\infty} \le C_0$ and $u(0) = 1$. Let

$$M(R) = \inf_{|x_0|=R} \sup_{\mathbb{B}_1(x_0)} |u(x)|$$

for R large. By using the result of (1.4), Bourgain and Kenig in [**BK**] showed that

$$M(R) \ge C\exp(-CR^{\frac{4}{3}}\log R),$$

where C depends on n and C_0. We can consider a similar quantitative unique continuation problem as (1.7) for higher order elliptic equations. Suppose that u is a solution to

$$(-\triangle)^m u = \overline{V} u \quad \text{in} \quad \mathbb{R}^n, \tag{1.8}$$

where $\|\overline{V}\|_{L^\infty} \leq 1$, $\|u\|_{L^\infty} \leq C_0$ and $u(0) = 1$. Theorem 2 implies that following corollary for higher order elliptic equations.

Corollary 1. *Let u be a solution to (1.8) and $n \geq 4m$. Then*

$$M(R) \geq C \exp(-CR^{2m} \log R),$$

where C depends on n, m and C_0.

An easy consequence of Theorem 2 is a strong unique continuation result for higher order elliptic equations when $n \geq 4m$. Due to the conclusion in Theorem 2, the solutions will not vanish of infinite order when $n \geq 4m$. Without using the conclusion of Theorem 2, we are also able to give a proof based on this variant of frequency function. We refer to, e.g. [**CG**], [**LSW**], for the strong unique continuation results of higher order elliptic equations by using Carleman estimates. Assume that

$$(-\triangle)^m u(x) = \overline{V}(x) u(x) \qquad \text{in } \Omega, \tag{1.9}$$

where $\overline{V}(x) \in L^\infty_{loc}(\Omega)$. A function $u \in L^2_{loc}(\Omega)$ is said to vanish of infinite order at some point $x_0 \in \Omega$ if for $R > 0$ sufficiently small,

$$\int_{\mathbb{B}_R(x_0)} u^2 \, dx = O(R^N) \tag{1.10}$$

for every positive integer N. We are able to establish the following theorem.

Theorem 3. *If u in (1.9) vanishes of infinite order at some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω.*

The outline of the paper is as follows. Section 2 is devoted to obtaining the vanishing order of Schrödinger equations. In Section 3, the vanishing order of higher order elliptic equations is shown. In section 4, we obtain the strong unique continuation for higher order elliptic equations. In the whole paper, we will use various letters, such as C, D, E, K, to denote the positive constants which may depend n and m, even if they are not explicitly stated. They may also vary from line to line. Especially the letters do not depend on V in section 2 and $\overline{V}$ in section 3.

2. Schrödinger equations

In this section, we focus on the maximal vanishing order of solutions in (1.3). Let $x_0 \in \mathbb{B}_1$. We define

$$H_{x_0}(r) = \int_{\mathbb{B}_r(x_0)} u^2 (r^2 - |x - x_0|^2)^\alpha \, dx. \tag{2.1}$$

The value of the constant $\alpha > 0$ will be determined later on. We can assume that $\mathbb{B}_r(x_0) \subset \mathbb{B}_{10}$ by choosing r suitable small. Without loss of generality, we may assume $x_0 = 0$ and denote $\mathbb{B}_r(0)$ as $\mathbb{B}_r$, that is,

$$H(r) = \int_{\mathbb{B}_r} u^2 (r^2 - |x|^2)^\alpha \, dx.$$

The advantage of the weight function $(r^2 - |x|^2)^\alpha$ in the integration is that the boundary term will not appear whenever we use divergence theorem. Moreover, the value of α will help reduce the order of vanishing. The function in (2.1) appeared in [**Ku2**] for the study of vortex of Ginzburg-Landau equations. We will also omit the integration on $\mathbb{B}_r$ when it is clear from the context. Taking the derivative with respect to r for $H(r)$, we get

$$H'(r) = 2\alpha r \int u^2 (r^2 - |x|^2)^{\alpha-1}\, dx.$$

Because of the presence of the weight function, as we mentioned before, there are no terms involving integration on the boundary. One of it's advantage is that it simplifies our calculations in the following. Furthermore,

$$\begin{aligned} H'(r) &= \frac{2\alpha}{r} \int u^2 (r^2 - |x|^2)^\alpha\, dx + \frac{2\alpha}{r} \int u^2 (r^2 - |x|^2)^{\alpha-1} |x|^2\, dx \\ &= \frac{2\alpha}{r} H(r) - \frac{1}{r} \int u^2 x \cdot \nabla (r^2 - |x|^2)^\alpha\, dx. \end{aligned}$$

Applying the divergence theorem for the second term in the right hand side of the latter equality, we get

$$H'(r) = \frac{2\alpha + n}{r} H(r) + \frac{1}{(\alpha+1) r} I(r), \tag{2.2}$$

where

$$I(r) = 2(\alpha+1) \int (x \cdot \nabla u) u (r^2 - |x|^2)^\alpha\, dx. \tag{2.3}$$

Using the divergence theorem again for $I(r)$, it follows that

$$\begin{aligned} I(r) &= -\int u \nabla u \cdot \nabla (r^2 - |x|^2)^{\alpha+1}\, dx \\ &= \int |\nabla u|^2 (r^2 - |x|^2)^{\alpha+1}\, dx + \int V u^2 (r^2 - |x|^2)^{\alpha+1}\, dx, \end{aligned} \tag{2.4}$$

where we perform integration by parts and use the equation (1.3) in the last equality.

We define our variant of frequency function as

$$N(r) = \frac{I(r)}{H(r)}. \tag{2.5}$$

Next we are going to study the monotonicity property of this special type of frequency function $N(r)$. We are able to obtain the following result.

Lemma 1. *There exists a constant C depending only on n such that*

$$N(r) + C \|V\|_{W^{1,\infty}} r^2$$

is nondecreasing function of $r \in (0,1)$.

Proof. To consider the monotonicity of $N(r)$, we shall consider the derivative of $I(r)$. By taking the derivative for $I(r)$ in (2.4) with respect to r,

$$I'(r) = 2(\alpha+1) r \int |\nabla u|^2 (r^2 - |x|^2)^\alpha\, dx + 2(\alpha+1) r \int V u^2 (r^2 - |x|^2)^\alpha\, dx.$$

We simply the first term in the right hand side of the latter equality. It yields that

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)}{r}\int |\nabla u|^2 (r^2-|x|^2)^{\alpha+1}\,dx - \frac{1}{r}\int x\cdot\nabla (r^2-|x|^2)^{\alpha+1}|\nabla u|^2\,dx \\ &\quad +2(\alpha+1)r\int V u^2 (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

Integrating by parts for the second term in the right hand side of the last equality gives that

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)+n}{r}\int |\nabla u|^2 (r^2-|x|^2)^{\alpha+1}\,dx + \sum_{j,l=1}^{n}\frac{2}{r}\int \partial_j u\partial_{jl}u x_l (r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +2(\alpha+1)r\int u^2 V (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

We do further integration by parts for the second term in the right hand side of the last inequality with respect to jth derivative. It follows that

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)+n}{r}\int |\nabla u|^2 (r^2-|x|^2)^{\alpha+1}\,dx - \frac{2}{r}\int \triangle u(\nabla u\cdot x)(r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad -\frac{2}{r}\int |\nabla u|^2 (r^2-|x|^2)^{\alpha+1}\,dx + \frac{4(\alpha+1)}{r}\int (\nabla u\cdot x)^2 (r^2-|x|^2)^{\alpha}\,dx \\ &\quad +2(\alpha+1)r\int V u^2 (r^2-|x|^2)^{\alpha}\,dx \\ &= \frac{2\alpha+n}{r}\int |\nabla u|^2 (r^2-|x|^2)^{\alpha+1}\,dx - \frac{2}{r}\int V u(\nabla u\cdot x)(r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +\frac{4(\alpha+1)}{r}\int (\nabla u\cdot x)^2 (r^2-|x|^2)^{\alpha}\,dx + 2(\alpha+1)r\int V u^2 (r^2-|x|^2)^{\alpha}\,dx, \end{aligned}$$

where we have used the equation (1.3) in the latter equality. We want to interpret the first term in the the right hand side of the last equality in terms of $I(r)$. In view of (2.4), we have

$$\begin{aligned} I'(r) &= \frac{2\alpha+n}{r}I(r) - \frac{2\alpha+n}{r}\int V u^2 (r^2-|x|^2)^{\alpha+1}\,dx - \frac{2}{r}\int V u(\nabla u\cdot x)(r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +\frac{4(\alpha+1)}{r}\int (\nabla u\cdot x)^2 (r^2-|x|^2)^{\alpha}\,dx + 2(\alpha+1)r\int V u^2 (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

We breaks down the second term in the last equality as

$$\begin{aligned} \frac{2\alpha+n}{r}\int V u^2 (r^2-|x|^2)^{\alpha+1}\,dx &= (2\alpha+n)r\int V u^2 (r^2-|x|^2)^{\alpha}\,dx \\ &\quad -\frac{2\alpha+n}{r}\int V u^2 (r^2-|x|^2)^{\alpha}|x|^2\,dx. \end{aligned}$$

Substituting the latter equality to $I'(r)$, one obtains

$$\begin{aligned} I'(r) &= \frac{2\alpha+n}{r}I(r) + (2-n)r\int V u^2 (r^2-|x|^2)^{\alpha}\,dx + \frac{2\alpha+n}{r}\int V u^2 (r^2-|x|^2)^{\alpha}|x|^2\,dx \\ &\quad +\frac{4(\alpha+1)}{r}\int (\nabla u\cdot x)^2 (r^2-|x|^2)^{\alpha}\,dx - \frac{2}{r}\int V u(\nabla u\cdot x)(r^2-|x|^2)^{\alpha+1}\,dx. \end{aligned}$$

Applying the divergence theorem for the last term in the right hand side of the latter equality and considering the fact that $V \in W^{1,\infty}$, we arrive at

$$\begin{aligned} I'(r) &= \frac{2\alpha+n}{r} I(r) + (2-n)r \int V u^2 (r^2-|x|^2)^\alpha \, dx + \frac{2\alpha+n}{r} \int V u^2 (r^2-|x|^2)^\alpha |x|^2 \, dx \\ &+ \frac{4(\alpha+1)}{r} \int (\nabla u \cdot x)^2 (r^2-|x|^2)^\alpha \, dx + \frac{1}{r} \int (\nabla V \cdot x) u^2 (r^2-|x|^2)^{\alpha+1} \, dx \\ &+ \frac{n}{r} \int V u^2 (r^2-|x|^2)^{\alpha+1} \, dx - \frac{2(\alpha+1)}{r} \int V u^2 (r^2-|x|^2)^\alpha |x|^2 \, dx. \end{aligned}$$

Combining the third term and seventh term in the right hand side of the latter equality gives that

$$\begin{aligned} I'(r) &\geq \frac{2\alpha+n}{r} I(r) + (2-n)r \int V u^2 (r^2-|x|^2)^\alpha \, dx + \frac{n-2}{r} \int V u^2 (r^2-|x|^2)^\alpha |x|^2 \, dx \\ &+ \frac{4(\alpha+1)}{r} \int (\nabla u \cdot x)^2 (r^2-|x|^2)^\alpha \, dx + \frac{1}{r} \int (\nabla V \cdot x) u^2 (r^2-|x|^2)^{\alpha+1} \, dx \\ &+ \frac{n}{r} \int V u^2 (r^2-|x|^2)^{\alpha+1} \, dx. \end{aligned}$$

By the definition of $H(r)$ in (2.1) and the assumption that $0 < r < 1$, we obtain

$$I'(r) \geq \frac{2\alpha+n}{r} I(r) - (3n+5)r \|V\|_{W^{1,\infty}} H(r) + \frac{4(\alpha+1)}{r} \int (\nabla u \cdot x)^2 (r^2-|x|^2)^\alpha \, dx. \tag{2.6}$$

In order to find the monotonicity of $N(r)$, it suffices to take the derivative for $N(r)$ with respect to r. Taking $H'(r)$ in (2.2) and $I'(r)$ in (2.6) into consideration, we get

$$\begin{aligned} N'(r) &= \frac{I'(r)H(r) - H'(r)I(r)}{H^2(r)} \\ &\geq 1/H^2(r) \big\{ \frac{2\alpha+n}{r} I(r)H(r) - (3n+5)r\|V\|_{W^{1,\infty}} H^2(r) \\ &+ \frac{4(\alpha+1)}{r} \int (\nabla u \cdot x)^2 (r^2-|x|^2)^\alpha \, dx \int u^2 (r-|x|^2)^\alpha \, dx - \frac{2\alpha+n}{r} I(r)H(r) \\ &- \frac{1}{r(\alpha+1)} I^2(r) \big\} \\ &\geq 1/H^2(r) \big\{ -(3n+5)r\|V\|_{W^{1,\infty}} H^2(r) - \frac{4(\alpha+1)}{r} \big(\int (x \cdot \nabla u) u (r^2-|x|^2)^\alpha \, dx \big)^2 \\ &+ \frac{4(\alpha+1)}{r} \int (\nabla u \cdot x)^2 (r^2-|x|^2)^\alpha \, dx \int u^2 (r-|x|^2)^\alpha \, dx \big\}, \end{aligned}$$

where we have used $I(r)$ in (2.3) in the last inequality. By Cauchy-Schwarz inequality, we know

$$\big(\int (x \cdot \nabla u) u (r^2-|x|^2)^\alpha \, dx \big)^2 \leq \int (\nabla u \cdot x)^2 (r^2-|x|^2)^\alpha \, dx \int u^2 (r-|x|^2)^\alpha \, dx.$$

We finally arrive at

$$N'(r) \geq -(3n+5)r\|V\|_{W^{1,\infty}},$$

which implies the conclusion in the lemma. $\square$

Let us compare more about the our variant of frequency function and that in [**GL**]. Both lead to monotonicity property. Unlike the monotonicity results in [**GL**], the function $V(x)$ is moved away from the exponential in Lemma 1 . Our monotonicity result only relies on the polynomial growth of $V(x)$. More important, the positive position C and the radius r do not depend on V in Lemma 1. The fact that r is independent of V is crucial in the propagation of smallness arguments in the proof of Theorem 1. With the help of monotonicity of $N(r)$, we are going to establish a L^2-version of Hadamard's three-ball theorem. For the variants of Hadamard's three-ball theorem, see e.g. [**JL**] and [**Ku**]. We also want to get rid of the weight function $(r^2-|x|^2)^\alpha$ in our function $H(r)$. In this process, the value of α helps reduce the coefficient in the following three-ball theorem, which provides better vanishing order. This is another advantage we introduce the weight function. Let

$$h(r)=\int_{\mathbb{B}_r(x_0)} u^2\,dx.$$

Without loss of generality, we may assume $x_0=0$. We can easily check that

$$H(r)\leq r^{2\alpha}h(r) \tag{2.7}$$

and

$$h(r)\leq \frac{H(\rho)}{(\rho^2-r^2)^\alpha} \tag{2.8}$$

for any $0<r<\rho<1$. We are able to obtain the following three-ball theorem.

Lemma 2. *Let $0<r_1<r_2<2r_2<r_3<1$. Then*

$$h(r_2)\leq \exp{(C\sqrt{M})}h^{\frac{\alpha_0}{\alpha_0+\beta_0}}(r_1)h^{\frac{\beta_0}{\alpha_0+\beta_0}}(r_3), \tag{2.9}$$

where

$$\alpha_0=\log\frac{r_3}{2r_2}$$

and

$$\beta_0=\log\frac{2r_2}{r_1}.$$

Proof. From (2.2), we have

$$\frac{H'(r)}{H(r)}=\frac{2\alpha+n}{r}+\frac{1}{(\alpha+1)r}N(r). \tag{2.10}$$

Taking integration from $2r_2$ to r_3 in the last identity gives that

$$\log\frac{H(r_3)}{H(2r_2)}=(2\alpha+n)\log\frac{r_3}{2r_2}+\frac{1}{\alpha+1}\int_{2r_2}^{r_3}\frac{N(r)}{r}\,dx. \tag{2.11}$$

By the monotonicity result in Lemma 1, it follows that

$$\log\frac{H(r_3)}{H(2r_2)}\geq (2\alpha+n)\log\frac{r_3}{2r_2}+\frac{1}{\alpha+1}(N(2r_2)+C\|V\|_{W^{1,\infty}}r_2^2)\log\frac{r_3}{2r_2}-\frac{C\|V\|_{W^{1,\infty}}}{\alpha+1}r_3^2,$$

that is,

$$\frac{\log\frac{H(r_3)}{H(2r_2)}+\frac{C\|V\|_{W^{1,\infty}}}{\alpha+1}r_3^2}{\log\frac{r_3}{2r_2}}\geq (2\alpha+n)+\frac{1}{\alpha+1}(N(2r_2)+C\|V\|_{W^{1,\infty}}r_2^2). \tag{2.12}$$

If we perform similar calculations on (2.10) by integrating from r_1 to $2r_2$, we deduce that

$$\begin{aligned}\log\frac{H(2r_2)}{H(r_1)} &= (2\alpha+n)\log\frac{2r_2}{r_1}+\frac{1}{\alpha+1}\int_{r_1}^{2r_2}\frac{N(r)}{r}\,dr\\ &\leq (2\alpha+n)\log\frac{2r_2}{r_1}+\frac{1}{\alpha+1}\log\frac{2r_2}{r_1}(N(2r_2)+C\|V\|_{W^{1,\infty}}r_2^2).\end{aligned}$$

Namely,

$$\frac{\log\frac{H(2r_2)}{H(r_1)}}{\log\frac{2r_2}{r_1}}\leq(2\alpha+n)+\frac{1}{\alpha+1}(N(2r_2)+C\|V\|_{W^{1,\infty}}r_2^2) \tag{2.13}$$

Combining the inequalities (2.12) and (2.13), note that $\|V\|_{W^{1,\infty}}\leq M$, we conclude that

$$\frac{\log\frac{H(r_3)}{H(2r_2)}+\frac{CM}{\alpha+1}r_3^2}{\log\frac{r_3}{2r_2}}\geq\frac{\log\frac{H(2r_2)}{H(r_1)}}{\log\frac{2r_2}{r_1}}. \tag{2.14}$$

Thanks to (2.7) and (2.8), we have

$$\begin{aligned}\log\frac{H(r_3)}{H(2r_2)} &\leq \log(r_3^{2\alpha}h(r_3))-\log(3r_2)^{\alpha}-\log h(r_2)\\ &\leq 2\alpha\log r_3+\log h(r_3)-2\alpha\log(2r_2)-\log h(r_2)+\alpha\log\frac{4}{3}.\end{aligned}$$

Therefore,

$$\frac{\log\frac{H(r_3)}{H(2r_2)}+\frac{CM}{\alpha+1}r_3^2}{\log\frac{r_3}{2r_2}}\leq 2\alpha+\frac{\log\frac{h(r_3)}{h(r_2)}}{\log\frac{r_3}{2r_2}}+\frac{\alpha\log\frac{4}{3}}{\log\frac{r_3}{2r_2}}+\frac{\frac{CM}{\alpha+1}r_3^2}{\log\frac{r_3}{2r_2}}. \tag{2.15}$$

We conduct the similar calculations as above for $\log\frac{H(2r_2)}{H(r_1)}$ in (2.13). Using (2.7) and (2.8) again,

$$\begin{aligned}\log\frac{H(2r_2)}{H(r_1)} &\geq \log((3r_2^2)^{\alpha}h(r_2))-\log r_1^{2\alpha}-\log h(r_1)\\ &\geq 2\alpha\log(2r_2)-\alpha\log\frac{4}{3}+\log h(r_2)-2\alpha\log r_1-\log h(r_1).\end{aligned}$$

So we obtain that

$$\frac{\log\frac{H(2r_2)}{H(r_1)}}{\log\frac{2r_2}{r_1}}\geq 2\alpha-\frac{\alpha\log\frac{4}{3}}{\log\frac{2r_2}{r_1}}+\frac{\log\frac{h(r_2)}{h(r_1)}}{\log\frac{2r_2}{r_1}}. \tag{2.16}$$

Taking (2.14), (2.15) and (2.16) into account, we get

$$\frac{\log\frac{h(r_3)}{h(r_2)}}{\log\frac{r_3}{2r_2}}+\frac{\alpha\log\frac{4}{3}}{\log\frac{r_3}{2r_2}}+\frac{\frac{CM}{\alpha+1}r_3^2}{\log\frac{r_3}{2r_2}}\geq-\frac{\alpha\log\frac{4}{3}}{\log\frac{2r_2}{r_1}}+\frac{\log\frac{h(r_2)}{h(r_1)}}{\log\frac{2r_2}{r_1}}.$$

Namely,

$$(\alpha_0+\beta_0)\alpha\log\frac{4}{3}+\beta_0\big(\log\frac{h(r_3)}{h(r_2)}+\frac{CM}{\alpha+1}r_3^2\big)\geq\alpha_0\log\frac{h(r_2)}{h(r_1)}.$$

Taking exponentials of both sides implies that

$$h(r_2)\leq\exp(C(\alpha+\frac{M}{\alpha+1}r_3^2))h^{\frac{\alpha_0}{\alpha_0+\beta_0}}(r_1)h^{\frac{\beta_0}{\alpha_0+\beta_0}}(r_3).$$

Note that $0 < r_3 < 1$. As we know, the minimum value of the exponential function in the last inequality will be achieved if we take $\alpha = \sqrt{M}$. Hence

$$h(r_2) \leq \exp{(C\sqrt{M})} h^{\frac{\alpha_0}{\alpha_0+\beta_0}}(r_1) h^{\frac{\beta_0}{\alpha_0+\beta_0}}(r_3),$$

where C is a constant depending only on n. We are done with the L^2-version of three-ball theorem. $\square$

From the above lemma, one can see that the appearance of α reduces the exponent of exponential in the L^2-version of three-ball theorem. Thanks to Lemma 2, we are able to establish a L^∞-version of three-ball theorem, which will be used in the propagation of smallness argument.

LEMMA 3. *Let* $0 < r_1 < r_2 < 2r_2 < r_3 < 1$. *Then*

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq C \exp{(C\sqrt{M})} (\frac{r_3^2}{r_3 - 2r_2})^{\frac{n}{2}} \|u\|^{\frac{\alpha_1}{\alpha_1+\beta_1}}_{L^\infty(\mathbb{B}_{r_1})} \|u\|^{\frac{\beta_1}{\alpha_1+\beta_1}}_{L^\infty(\mathbb{B}_{r_3})}, \tag{2.17}$$

where

$$\alpha_1 = \log \frac{r_3}{\frac{2}{3}(r_2 + r_3)}$$

and

$$\beta_1 = \log \frac{\frac{2}{3}(r_2 + r_3)}{r_1}.$$

PROOF. Using the standard elliptic theory for the solution in (1.3), we have

$$\|u\|_{L^\infty(\mathbb{B}_\delta)} \leq C(\|V\|_{L^\infty} + 1)^{\frac{n}{2}} \delta^{-\frac{n}{2}} \|u\|_{L^2(\mathbb{B}_{2\delta})}, \tag{2.18}$$

here C does not depends on δ. By some rescaling argument,

$$\|u\|_{L^\infty(\mathbb{B}_r)} \leq C(\|V\|_{L^\infty} + 1)^{\frac{n}{2}} (\rho - r)^{-\frac{n}{2}} \|u\|_{L^2(\mathbb{B}_\rho)}$$

for $0 < r < \rho < 1$. Then

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq C(\|V\|_{L^\infty} + 1)^{\frac{n}{2}} (r_3 - 2r_2)^{-\frac{n}{2}} \|u\|_{L^2(\mathbb{B}_{\frac{r_2+r_3}{3}})}.$$

Taking advantage of Lemma 2, we deduce that

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq C(\|V\|_{L^\infty} + 1)^{\frac{n}{2}} (r_3 - 2r_2)^{-\frac{n}{2}} r_3^n \exp{(C\sqrt{M})} \|u\|^{\frac{\alpha_1}{\alpha_1+\beta_1}}_{L^\infty(\mathbb{B}(r_1))} \|u\|^{\frac{\beta_1}{\alpha_1+\beta_1}}_{L^\infty(\mathbb{B}(r_3))}.$$

Recall that $\|V\|_{W^{1,\infty}} \leq M$. Thus, we arrive at the conclusion. $\square$

Now we are ready to prove Theorem 1. We apply the idea of propagation of smallness which is based on overlapping of three-ball argument. Similar arguments have been employed in [**DF**].

PROOF OF THEOREM 1. We choose a small r such that

$$\sup_{\mathbb{B}_{\frac{r}{2}(0)}} |u| = \epsilon.$$

Obviously, $\epsilon > 0$. Since $\sup_{|x|\leq 1} |u(x)| \geq 1$, there exists some $\bar{x} \in \mathbb{B}_1$ such that $u(\bar{x}) = \sup_{|x|\leq 1} |u(x)| \geq 1$. We select a sequence of balls with radius r centered at $x_0 = 0,\ x_1, \cdots, x_d$ so that $x_{i+1} \in \mathbb{B}_{\frac{r}{2}}(x_i)$ and $\bar{x} \in \mathbb{B}_r(x_d)$, where d depends on the radius r

which we will fix later on. Employing Lemma 3 with $r_1 = \frac{r}{2}$, $r_2 = r$, and $r_3 = 3r$ and the boundedness assumption of u, we get

$$\|u\|_{L^\infty(\mathbb{B}_r)} \le C_1 \epsilon^\theta \exp{(C\sqrt{M})}$$

where $1 < \theta = \frac{\log \frac{9}{8}}{\log 6} < 1$ and C_1 depends on the L^∞-norm of u.

Iterating the above argument with Lemma 3 for balls centered at x_i and using the fact that $\|u\|_{L^\infty(\mathbb{B}_{\frac{r}{2}}(x_{i+1}))} \le \|u\|_{L^\infty(\mathbb{B}_r(x_i))}$, we have

$$\|u\|_{L^\infty(\mathbb{B}_r(x_i))} \le C_i \epsilon^{D_i} \exp{(E_i\sqrt{M})}$$

for $i = 0, 1, \cdots, d$, where C_i is a constant depending on d and L^∞-norm of u, and D_i, E_i are constants depending on d. By the fact that $u(\bar{x}) \ge 1$ and $\bar{x} \in \mathbb{B}_r(x_d)$, we obtain

$$\sup_{\mathbb{B}_{\frac{r}{2}}(0)} |u| = \epsilon \ge K_1 \exp{(-K_2\sqrt{M})},$$

where K_1 is a constant depending on d and L^∞-norm of u, and K_2 is a constant depending on d.

Applying the L^∞ type of three-ball lemma again centered at origin again with $r_2 = \frac{r}{2} > r_1$ and $r_3 = 3r$, where r_1 is sufficiently small, we have

$$K_1 \exp{(-K_2\sqrt{M})} \le C \exp{(C\sqrt{M})} \|u\|^{\frac{\alpha_1}{\alpha_1+\beta_1}}_{L^\infty(\mathbb{B}_{r_1})} C_0^{\frac{\beta_1}{\alpha_1+\beta_1}}.$$

Recall that C_0 is the L^∞ norm of u in $\mathbb{B}_{10}$. Then

$$K_3^{1+q} \exp{(-(1+q)K_4\sqrt{M})} \le \|u\|_{L^\infty(\mathbb{B}_{r_1})}, \tag{2.19}$$

where K_3 depends on d and L^∞ norm of u, K_4 depends on d, and

$$q = \frac{\beta_1}{\alpha_1} = \frac{\log \frac{7}{3}r - \log r_1}{\log \frac{9}{7}} = -K_5 + \log\frac{9}{7} \log\frac{1}{r_1}$$

with constant $K_5 > 0$ depending on r. Now we can fix the small r. For instance, let $r = \frac{1}{100}$. Then the number d is also determined. The inequality (2.19) implies that

$$\|u\|_{L^\infty(\mathbb{B}_{r_1})} \ge K_6 r_1^{K_7\sqrt{M}},$$

where the constants K_6, K_7 depend on the dimension n and C_0. Therefore, Theorem 1 is completed. □

3. Higher order elliptic equations

In this section, we consider the vanishing order of solutions for the higher order elliptic equations. As far as we know, the explicit vanishing order seems to be unknown in the literature. Due to the complexity of its structure, we decompose the model in (1.6) into a system of m semilinear equations, that is,

$$\begin{cases} -\triangle u_1 &= u_2, \\ -\triangle u_i &= u_{i+1}, \quad i = 2, \cdots, m-1, \\ -\triangle u_m &= \overline{V} u_1. \end{cases} \tag{3.1}$$

Note that $u_1 = u$. Inspired by our frequency function in section 2, it is nature to consider the following function for the system of semilinear equations in (3.1). Let

$$H_{x_0}(r) = \sum_{i=1}^{m} \int_{\mathbb{B}_r(x_0)} u_i^2 (r^2 - |x - x_0|^2)^\alpha \, dx. \tag{3.2}$$

As before, we may assume $x_0 = 0$ and omit the integration on $\mathbb{B}_r$ if it is clear from the context. Namely,

$$H(r) = \sum_{i=1}^{m} \int u_i^2 (r^2 - |x|^2)^\alpha \, dx.$$

The value of the constant $\alpha > 0$ will be determined later on. If one takes derivative for $H(r)$ with respect to r, following the similar calculations in section 2, one has

$$\begin{aligned} H'(r) &= 2\alpha r \sum_{i=1}^{m} \int u_i^2 (r^2 - |x|^2)^{\alpha-1} \, dx \\ &= \frac{2\alpha}{r} \sum_{i=1}^{m} \int u_i^2 (r^2 - |x|^2)^\alpha \, dx + \frac{2\alpha}{r} \sum_{i=1}^{m} \int u_i^2 (r^2 - |x|^2)^{\alpha-1} |x|^2 \, dx \\ &= \frac{2\alpha}{r} H(r) - \frac{1}{r} \sum_{i=1}^{m} \int u_i^2 x \cdot \nabla (r^2 - |x|^2)^\alpha \, dx. \end{aligned} \tag{3.3}$$

Performing the divergence theorem for the second term in the right hand side of the last equality, we obtain that

$$H'(r) = \frac{2\alpha + n}{r} H(r) + \frac{1}{(\alpha+1)r} I(r), \tag{3.4}$$

where

$$I(r) = 2(\alpha+1) \sum_{i=1}^{m} \int (x \cdot \nabla u_i) u_i (r^2 - |x|^2)^\alpha \, dx. \tag{3.5}$$

Applying the divergence theorem on $I(r)$, we have

$$\begin{aligned} I(r) &= -\sum_{i=1}^{m} \int \nabla u_i \cdot \nabla (r^2 - |x|^2)^{\alpha+1} u_i \, dx \\ &= \sum_{i=1}^{m} \int |\nabla u_i|^2 (r^2 - |x|^2)^{\alpha+1} \, dx + \sum_{i=1}^{m} \int \triangle u_i u_i (r^2 - |x|^2)^{\alpha+1} \, dx. \end{aligned} \tag{3.6}$$

Considering the systems of equations (3.1), it follows that

$$\begin{aligned} I(r) &= \sum_{i=1}^{m} \int |\nabla u_i|^2 (r^2 - |x|^2)^{\alpha+1} \, dx - \sum_{i=1}^{m-1} \int u_{i+1} u_i (r^2 - |x|^2)^{\alpha+1} \, dx \\ &\quad - \int \overline{V} u_m u_1 (r^2 - |x|^2)^{\alpha+1} \, dx. \end{aligned} \tag{3.7}$$

For the higher order elliptic equations, we define our variant of frequency function as

$$N(r) = \frac{I(r)}{H(r)}. \tag{3.8}$$

Since we are dealing with more complex structure, more careful calculations are devoted. Different from the semilinear equation case, higher regularity, i.e. $\overline{V}(x) \in W^{1,\infty}$ seems not be helpful. We consider the case that $\overline{V}(x) \in L^\infty$. We are able to obtain the following the monotonicity property for the frequency function $N(r)$.

LEMMA 4. *There exists a constant C depending only on n, m such that*

$$\exp(Cr)\big(N(r) + \alpha(\|\overline{V}\|_{L^\infty} + 1) + (\|\overline{V}\|_{L^\infty} + 1)^2\big)$$

is nondecreasing function of $r \in (0,1)$.

PROOF. To obtain the monotonicity result, we shall consider the derivative of $I(r)$. Now differentiating $I(r)$ in (3.6) with respect to r,

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)}{r}\sum_{i=1}^m \int |\nabla u_i|^2 (r^2-|x|^2)^{\alpha+1}\,dx - \frac{1}{r}\sum_{i=1}^m \int |\nabla u_i|^2 \nabla(r^2-|x|^2)^{\alpha+1}\cdot x\,dx \\ &\quad +2(\alpha+1)r\sum_{i=1}^m \int \triangle u_i u_i (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

Integrating by parts for the second term in the right hand side of the latter equality,

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)+n}{r}\sum_{i=1}^m \int |\nabla u_i|^2 (r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +\frac{2}{r}\sum_{i=1}^m\sum_{l=1}^n \int \partial_{jl}u_i\partial_j u_i x_l (r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +2(\alpha+1)r\sum_{i=1}^m \int \triangle u_i u_i (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

If one performs the divergence theorem with respect to jth derivative on the second term in the right hand side of the last inequality, one has

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)+n}{r}\sum_{i=1}^m \int |\nabla u_i|^2 (r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad -\frac{2}{r}\sum_{i=1}^m \int (\nabla u_i\cdot x)\triangle u_i (r^2-|x|^2)^{\alpha+1}\,dx - \frac{2}{r}\sum_{i=1}^m \int |\nabla u_i|^2 (r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +\frac{4(\alpha+1)}{r}\sum_{i=1}^m \int (\nabla u_i\cdot x)^2 (r^2-|x|^2)^{\alpha}\,dx + 2(\alpha+1)r\sum_{i=1}^m \int \triangle u_i u_i (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

Using the equivalent system of equations in (3.1), it follows that

$$\begin{aligned} I'(r) &= \frac{2\alpha+n}{r}\sum_{i=1}^m \int |\nabla u_i|^2 (r^2-|x|^2)^{\alpha+1}\,dx + \frac{2}{r}\sum_{i=1}^{m-1} \int (\nabla u_i\cdot x) u_{i+1} (r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad +\frac{2}{r}\int (\nabla u_m\cdot x)\overline{V} u_1 (r^2-|x|^2)^{\alpha+1}\,dx + \frac{4(\alpha+1)}{r}\sum_{i=1}^m \int (\nabla u_i\cdot x)^2 (r^2-|x|^2)^{\alpha}\,dx \\ &\quad -2(\alpha+1)r\sum_{i=1}^{m-1} \int u_{i+1}u_i (r^2-|x|^2)^{\alpha}\,dx - 2(\alpha+1)r\int \overline{V} u_1 u_m (r^2-|x|^2)^{\alpha}\,dx. \end{aligned}$$

We want to transform the first term in the right hand side of the latter inequality in term of $I(r)$. Taking (3.7) into consideration and performing some calculations, we have

$$\begin{aligned} I'(r) &= \frac{2\alpha+n}{r}I(r)+(n-2)r\sum_{i=1}^{m-1}\int u_{i+1}u_i(r^2-|x|^2)^\alpha\,dx \\ &+(n-2)r\int \overline{V}u_m u_1(r^2-|x|^2)^\alpha\,dx-\frac{2\alpha+n}{r}\sum_{i=1}^{m-1}\int u_{i+1}u_i(r^2-|x|^2)^\alpha|x|^2\,dx \\ &-\frac{2\alpha+n}{r}\int \overline{V}u_m u_1(r^2-|x|^2)^\alpha|x|^2\,dx+\frac{4(\alpha+1)}{r}\sum_{i=1}^{m}\int(\nabla u_i\cdot x)^2(r^2-|x|^2)^\alpha\,dx \\ &+\frac{2}{r}\sum_{i=1}^{m-1}\int(\nabla u_i\cdot x)u_{i+1}(r^2-|x|^2)^{\alpha+1}\,dx+\frac{2}{r}\int(\nabla u_m\cdot x)\overline{V}u_1(r^2-|x|^2)^{\alpha+1}\,dx. \end{aligned}$$

Now we estimate each term in the right hand side of the last equality. Using Hölder's inequality and the definition of $H(r)$ in (3.2), we obtain

$$\begin{aligned} (n-2)r\sum_{i=1}^{m-1}\int u_{i+1}u_i(r^2-|x|^2)^\alpha\,dx &+ (n-2)r\int\overline{V}u_m u_1(r^2-|x|^2)^\alpha\,dx \\ &\geq -Cr(\|\overline{V}\|_{L^\infty}+1)H(r) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} -\frac{2\alpha+n}{r}\sum_{i=1}^{m-1}\int u_{i+1}u_i(r^2-|x|^2)^\alpha|x|^2\,dx &- \frac{2\alpha+n}{r}\int\overline{V}u_m u_1(r^2-|x|^2)^\alpha|x|^2\,dx \\ &\geq -(2\alpha+n)r(\|\overline{V}\|_{L^\infty}+1)H(r). \end{aligned} \tag{3.10}$$

Similarly, by Hölder's inequality,

$$\begin{aligned} \frac{2}{r}\sum_{i=1}^{m-1}\int(\nabla u_i\cdot x)u_{i+1}(r^2-|x|^2)^{\alpha+1}\,dx &\geq -2\sum_{i=1}^{m-1}\int|\nabla u_i|^2(r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad -2\sum_{i=1}^{m-1}\int u_{i+1}^2(r^2-|x|^2)^{\alpha+1}\,dx \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \frac{2}{r}\int(\nabla u_m\cdot x)\overline{V}u_1(r^2-|x|^2)^{\alpha+1}\,dx &\geq -2\int|\nabla u_m|^2(r^2-|x|^2)^{\alpha+1}\,dx \\ &\quad -\|\overline{V}\|_{L^\infty}^2\int u_1^2(r^2-|x|^2)^{\alpha+1}\,dx. \end{aligned} \tag{3.12}$$

For the ease of the notation, let

$$v=\|\overline{V}\|_{L^\infty}+1.$$

Combining the inequalities (3.11) and (3.12) and taking (3.7) into account, we get

$$
\begin{aligned}
\frac{2}{r}\int \big(\sum_{i=1}^{m-1}(\nabla u_i\cdot x)u_{i+1} &+ (x\cdot\nabla u_m)\overline{V}u_1\big)(r^2-|x|^2)^{\alpha+1}\,dx \\
&\geq -C\sum_{i=1}^{m}\int |\nabla u_i|^2(r^2-|x|^2)^{\alpha+1}\,dx \\
&\quad -Cv^2\sum_{i=1}^{m}\int |u_i|^2(r^2-|x|^2)^{\alpha+1}\,dx \\
&\geq -CI(r)-Cv^2rH(r) \\
&\quad -C\int \big(\sum_{i=1}^{m-1}u_{i+1}u_i+\overline{V}u_mu_1\big)(r^2-|x|^2)^{\alpha+1}\,dx \\
&\geq -CI(r)-Cv^2rH(r). \qquad (3.13)
\end{aligned}
$$

Therefore, together with (3.9), (3.10), and (3.13),

$$
\begin{aligned}
I'(r) \;\geq\; & \frac{2\alpha+n}{r}I(r)-CI(r)-C(\alpha v+v^2)H(r) \\
&+\frac{4(\alpha+1)}{r}\sum_{i=1}^{m}\int(\nabla u_i\cdot x)^2(r^2-|x|^2)^{\alpha}\,dx,
\end{aligned}
$$

where C depends only on n and m. In order to get monotonicity of the frequency function, we differentiate $N(r)$. Recall $H(r)$ in (3.2) and $H'(r)$ in (3.4).

$$
\begin{aligned}
N'(r) &= \frac{I'(r)H(r)-H'(r)I(r)}{H^2(r)} \\
&\geq \frac{1}{H^2(r)}\Big\{\frac{2\alpha+n}{r}I(r)H(r)-CI(r)H(r)-C(\alpha v+v^2)H^2(r) \\
&\quad +\frac{4(\alpha+1)}{r}\sum_{i=1}^{m}\int(\nabla u_i\cdot x)^2(r^2-|x|^2)^{\alpha}\,dx\sum_{i=1}^{m}\int|u_i|^2(r^2-|x|^2)^{\alpha}\,dx \\
&\quad -\frac{2\alpha+n}{r}I(r)H(r)-\frac{1}{(\alpha+1)r}I^2(r)\Big\} \\
&\geq \frac{1}{H^2(r)}\Big\{\frac{4(\alpha+1)}{r}\sum_{i=1}^{m}\int(\nabla u_i\cdot x)^2(r^2-|x|^2)^{\alpha}\,dx\sum_{i=1}^{m}\int|u_i|^2(r^2-|x|^2)^{\alpha}\,dx \\
&\quad -\frac{4(\alpha+1)}{r}\sum_{i=1}^{m}\int(\nabla u_i\cdot x)u_i(r^2-|x|^2)^{\alpha}\,dx-CI(r)H(r)-C(\alpha v+v^2)H^2(r)\Big\},
\end{aligned}
$$

where we have used $I(r)$ in (3.5) in the last inequality. By Cauchy-Schwartz inequality,

$$N'(r)+CN(r)+C(\alpha v+v^2)\geq 0.$$

Consequently,

$$\exp{(Cr)}(N(r)+\alpha v+v^2)\ \text{ is nondecreasing.}$$

We complete the proof of the lemma. □

As the conclusion in Lemma 4 indicates, the monotonicity property only relies on the polynomial growth of $\overline{V}$ and r does not depend on $\overline{V}$. We are going to establish a L^2-version of three-ball theorem. For convenience, let

$$\overline{N}(r) = \exp{(Cr)}(N(r) + \alpha v + v^2).$$

We also need to remove the weight function $(r^2 - |x|^2)^\alpha$ in $H(r)$. As in the section 2, let

$$h(r) = \sum_{i=1}^{m} \int_{B_r(x_0)} u_i^2 \, dx.$$

As usual, we will omit the dependent of the center of x_0 for the ball. It is easy to check that

$$H(r) \leq r^{2\alpha} h(r) \tag{3.14}$$

and

$$h(r) \leq \frac{H(\rho)}{(\rho^2 - r^2)^\alpha} \tag{3.15}$$

for any $0 < r < \rho < 1$.

Based on the monotonicity of $N(r)$ in the last lemma, we are able to establish the following L^2-type of three-ball theorem.

Lemma 5. *Let $0 < r_1 < r_2 < 2r_2 < r_3 < 1$. Then*

$$h(r_2) \leq (\frac{r_3}{2r_2})^{CM} \exp{(CM)} h^{\frac{\alpha_2}{\alpha_2+\beta_2}}(r_1) h^{\frac{\beta_2}{\alpha_2+\beta_2}}(r_3)$$

where

$$\alpha_2 = \log \frac{r_3}{2r_2}$$

and

$$\beta_2 = C \log \frac{2r_2}{r_1},$$

where C depends only on n and m.

Proof. From (3.4), we deduce that

$$\frac{H'(r)}{H(r)} = \frac{2\alpha + n}{r} + \frac{1}{(\alpha+1)r} N(r). \tag{3.16}$$

On one hand, integrating from r_1 to $2r_2$ on the equality (3.16) gives that

$$\begin{aligned}
\log \frac{H(2r_2)}{H(r_1)} &= (2\alpha + n) \log \frac{2r_2}{r_1} + \frac{1}{\alpha+1} \int_{r_1}^{2r_2} \frac{N(r)}{r} \, dr \\
&\leq (2\alpha + n) \log \frac{2r_2}{r_1} + \frac{\overline{N}(2r_2)}{\alpha+1} \log \frac{2r_2}{r_1} - \frac{\alpha v + v^2}{\alpha+1} \log \frac{2r_2}{r_1},
\end{aligned}$$

where we have used Lemma 4 in the last inequality. Namely,

$$\frac{\log H(2r_2)/H(r_1)}{\log 2r_2/r_1} - (2\alpha + n) + \frac{\alpha v + v^2}{\alpha+1} \leq \frac{\overline{N}(2r_2)}{\alpha+1}. \tag{3.17}$$

On the other hand, integrating from $2r_2$ to r_3 on the equality (3.16) implies that

$$\begin{aligned}\log\frac{H(r_3)}{H(2r_2)} &= (2\alpha+n)\log\frac{r_3}{2r_2}+\frac{1}{\alpha+1}\int_{2r_2}^{r_3}\frac{N(r)}{r}\,dr\\ &\geq (2\alpha+n)\log\frac{r_3}{2r_2}+\frac{1}{\alpha+1}\exp(-C)\overline{N}(2r_2)\log\frac{r_3}{2r_2}-\frac{\alpha v+v^2}{\alpha+1}\log\frac{r_3}{2r_2},\end{aligned}$$

that is,

$$\frac{\log H(r_3)/H(2r_2)}{\log r_3/2r_2}-(2\alpha+n)+\frac{\alpha v+v^2}{\alpha+1}\geq\frac{1}{\alpha+1}\exp(-C)\overline{N}(2r_2). \tag{3.18}$$

Taking (3.17) and (3.18) into considerations, we get

$$\frac{\log H(2r_2)/H(r_1)}{\log 2r_2/r_1}\leq C\frac{\log H(r_3)/H(2r_2)}{\log r_3/2r_2}+C\frac{\alpha v+v^2}{\alpha+1}. \tag{3.19}$$

Thanks to (3.14) and (3.15),

$$\begin{aligned}\log\frac{H(r_3)}{H(2r_2)} &\leq \log(r_3^{2\alpha}h(r_3))-\log(3r_2)^{\alpha}-\log h(r_2)\\ &\leq 2\alpha\log r_3+\log h(r_3)-2\alpha\log(2r_2)-\log h(r_2)+\alpha\log\frac{4}{3}.\end{aligned}$$

Therefore,

$$\frac{\log\frac{H(r_3)}{H(2r_2)}}{\log\frac{r_3}{2r_2}}+\frac{\alpha v+v^2}{\alpha+1}\leq\alpha+\frac{\log\frac{h(r_3)}{h(r_2)}}{\log\frac{r_3}{2r_2}}+\frac{\alpha\log\frac{4}{3}}{\log\frac{r_3}{2r_2}}+\frac{\alpha v+v^2}{\alpha+1}. \tag{3.20}$$

We do the similar calculations for $\log\frac{H(2r_2)}{H(r_1)}$. Using (3.14) and (3.15) again,

$$\begin{aligned}\log\frac{H(2r_2)}{H(r_1)} &\geq \log((3r_2^2)^{\alpha}h(r_2))-\log r_1^{2\alpha}-\log h(r_1)\\ &\geq 2\alpha\log(2r_2)-\alpha\log\frac{4}{3}+\log h(r_2)-2\alpha\log r_1-\log h(r_1).\end{aligned}$$

Thus,

$$\frac{\log\frac{H(2r_2)}{H(r_1)}}{\log\frac{2r_2}{r_1}}\geq 2\alpha-\frac{\alpha\log\frac{4}{3}}{\log\frac{2r_2}{r_1}}+\frac{\log\frac{h(r_2)}{h(r_1)}}{\log\frac{2r_2}{r_1}}. \tag{3.21}$$

Taking (3.19), (3.20) and (3.21) into account, we have

$$C\frac{\log\frac{h(r_3)}{h(r_2)}}{\log\frac{r_3}{2r_2}}+C\frac{\alpha\log\frac{4}{3}}{\log\frac{r_3}{2r_2}}+C(\alpha+\frac{\alpha v+v^2}{\alpha+1})\geq-\frac{\alpha\log\frac{4}{3}}{\log\frac{2r_2}{r_1}}+\frac{\log\frac{h(r_2)}{h(r_1)}}{\log\frac{2r_2}{r_1}}.$$

Namely,

$$(\alpha_2+\beta_2)\alpha\log\frac{4}{3}+\beta_2\log\frac{h(r_3)}{h(r_2)}+\alpha_2\beta_2(\alpha+\frac{\alpha v+v^2}{\alpha+1})\geq\alpha_2\log\frac{h(r_2)}{h(r_1)}.$$

Taking exponentials of both sides and performing some simplications, we obtain

$$h(r_2)\leq\exp\big(\frac{\alpha_2\beta_2}{\alpha_2+\beta_2}(\alpha+\frac{\alpha v+v^2}{\alpha+1})\big)\exp(\alpha)h^{\frac{\alpha_2}{\alpha_2+\beta_2}}(r_1)h^{\frac{\beta_2}{\alpha_2+\beta_2}}(r_3).$$

Since

$$\frac{\alpha_2\beta_2}{\alpha_2+\beta_2} \leq \log \frac{r_3}{2r_2},$$

we have

$$h(r_2) \leq (\frac{r_3}{2r_2})^{(\alpha+\frac{\alpha v+v^2}{\alpha+1})} \exp{(\alpha)} h^{\frac{\alpha_2}{\alpha_2+\beta_2}}(r_1) h^{\frac{\beta_2}{\alpha_2+\beta_2}}(r_3).$$

As we know, the minimum value of the function $\alpha+\frac{\alpha v+v^2}{\alpha+1}$ is achieved in the case of $\alpha = v$. Recall that $v = \|\overline{V}\|_{L^\infty}+1 \leq 2M$. Therefore, the lemma is completed.

□

Again we need to establish a L^∞-version of three-ball theorem. However, the classical elliptic estimates as (2.18) does not seem to be known for higher order elliptic equations in the literature. We will deduce a similar estimate by Sobolev inequality and a $W^{2m,p}$ type estimate. We first present a $W^{2m,p}$ type estimates for higher order elliptic equations (see e.g. [**LWZ**]). Let u satisfy the following equation

$$(-\triangle)^m u = g(x) \qquad \text{in } \mathbb{B}. \tag{3.22}$$

Then we have

Lemma 6. *Let $1 < p < \infty$. Suppose $u \in W^{2m,p}$ satisfies (3.22). Then there exits a constant $C > 0$ depending only on n, m such that for any $\sigma \in (0,1)$,*

$$\|u\|_{W^{2m,p}(\mathbb{B}_\sigma)} \leq \frac{C(n,m)}{(1-\sigma)^{2m}}(\|g\|_{L^p(\mathbb{B})}+\|u\|_{L^p(\mathbb{B})}). \tag{3.23}$$

Upon a rescaling argument, we have

$$\|u\|_{W^{2m,p}(\mathbb{B}_{\sigma R})} \leq \frac{C(n,m)}{(1-\sigma)^{2m}R^{2m}}(R^{2m}\|g\|_{L^p(\mathbb{B}_R)}+\|u\|_{L^p(\mathbb{B}_R)}) \tag{3.24}$$

for $0 < R < 1$.

Applying Lemma 6, we are able to establish the L^∞-version of three-ball theorem for the solutions in (1.6).

Lemma 7. *Let $0 < r_1 < r_2 < 4r_2 < r_3 < 1$ and $n \geq 4m$. Then*

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq C\exp{(CM)}(r_3-4r_2)^{-\frac{n}{2}}(\frac{3r_3}{2(2r_2+r_3)})^{CM}\|u\|^{\frac{\alpha_3}{\alpha_3+\beta_3}}_{L^\infty(\mathbb{B}_{r_1})}\|u\|^{\frac{\beta_3}{\alpha_3+\beta_3}}_{L^\infty(\mathbb{B}_{r_3})} \tag{3.25}$$

where

$$\alpha_3 = \log \frac{3r_3}{2(2r_2+r_3)}$$

and

$$\beta_3 = C\log\frac{2r_2+r_3}{3r_1}.$$

Proof. Thanks to Lemma 6 in the case of $p = 2$, we can estimate the solution in (1.6) by the following

$$\|u\|_{W^{2m,2}(\mathbb{B}_\sigma)} \leq \frac{C}{(1-\sigma)^{2m}}(\|\overline{V}\|_{L^\infty}+1)\|u\|_{L^2(\mathbb{B})}.$$

By Sobolev imbedding inequality, if $n > 4m$,

$$\|u\|_{L^{\frac{2n}{n-4m}}(\mathbb{B}_\sigma)} \leq \frac{C(\sigma)}{(1-\sigma)^{2m}}(\|\overline{V}\|_{L^\infty}+1)\|u\|_{L^2(\mathbb{B})},$$

where $C(\sigma)$ depends on σ, n and m. Applying Lemma 6 again with $p = \frac{2n}{n-4m}$ and the latter inequality, note that $p = \frac{2n}{n-4m} > 2$,

$$\begin{aligned}\|u\|_{W^{2m,\frac{2n}{n-4m}}(\mathbb{B}_{\sigma^2})} &\leq \frac{C(\sigma)}{(1-\sigma)^{2m}}(\|\overline{V}\|_{L^\infty}+1)\|u\|_{L^{\frac{2n}{n-4m}}(\mathbb{B}_\sigma)} \\ &\leq \frac{C(\sigma)}{(1-\sigma)^{4m}}(\|\overline{V}\|_{L^\infty}+1)^2\|u\|_{L^2(\mathbb{B})}.\end{aligned}$$

As we know

$$\|u\|_{W^{2m,q}(\mathbb{B})} \geq C\|u\|_{L^\infty(\mathbb{B})}, \quad \text{if} \quad q > \frac{n}{2m}.$$

Employing the above bootstrap argument finite times, e.g. k times, which depends only on n and m and using the above Sobolev imbedding inequality, we get

$$\|u\|_{L^\infty(\mathbb{B}_{\sigma^k})} \leq \frac{C(\sigma)}{(1-\sigma)^{2^k m}}(\|\overline{V}\|_{L^\infty}+1)^k\|u\|_{L^2(\mathbb{B})}.$$

Let $\sigma^k = \frac{1}{2}$,

$$\|u\|_{L^\infty(\mathbb{B}_{\frac{1}{2}})} \leq C(\|\overline{V}\|_{L^\infty}+1)^C\|u\|_{L^2(\mathbb{B})},$$

where C depends on only n and m. If $n = 4m$, we will have the similar result by applying the bootstrap arguments twice. By a rescaling argument, we have

$$\|u\|_{L^\infty(\mathbb{B}_\delta)} \leq C(\|\overline{V}\|_{L^\infty}+1)^C\delta^{-\frac{n}{2}}\|u\|_{L^2(\mathbb{B}_{2\delta})},$$

if $0 < \delta < \frac{1}{2}$. Furthermore, we get

$$\|u\|_{L^\infty(\mathbb{B}_r)} \leq Cv^C(\rho-r)^{-\frac{n}{2}}\|u\|_{L^2(\mathbb{B}_\rho)}$$

for $0 < r < \rho < \frac{1}{2}$. Recall that $v = (\|\overline{V}\|_{L^\infty}+1) \leq 2M$. Thus,

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq CM^C(r_3-2r_2)^{-\frac{n}{2}}\|u\|_{L^2(\mathbb{B}_{\frac{r_2+r_3}{3}})}.$$

Based on Lemma 5 and the latter inequality, we deduce that

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq C\exp(CM)(r_3-2r_2)^{-\frac{n}{2}}\left(\frac{3r_3}{2(r_2+r_3)}\right)^{CM}h^{\frac{\alpha'}{\alpha'+\beta'}}\left(\frac{r_1}{2}\right)h^{\frac{\beta'}{\alpha'+\beta'}}(r_3), \tag{3.26}$$

where

$$\alpha' = \log\frac{3r_3}{2(r_2+r_3)}$$

and

$$\beta' = C\log\frac{2(r_2+r_3)}{3r_1}.$$

Taking Lemma 6 and (3.24) into account with $p = 2$, we have

$$h\left(\frac{r_1}{2}\right) \leq C(\|\overline{V}\|_{L^\infty}+1)r_1^{-2m}\|u\|_{L^2(\mathbb{B}_{r_1})}$$

and

$$h(r_3) \leq C(\|\overline{V}\|_{L^\infty}+1)r_3^{-2m}\|u\|_{L^2(\mathbb{B}_{2r_3})}.$$

It is true that

$$r^{-2m}\|u\|_{L^2(\mathbb{B}_r)} \leq r^{n/2-2m}\|u\|_{L^\infty(\mathbb{B}_r)} \leq \|u\|_{L^\infty(\mathbb{B}_r)}$$

if $0 < r < 1$ and $n \geq 4m$. From (3.26) and the last three inequalities, we obtain

$$\|u\|_{L^\infty(\mathbb{B}_{r_2})} \leq C\exp{(CM)}(r_3-2r_2)^{-\frac{n}{2}}(\frac{3r_3}{2(r_2+r_3)})^{CM}\|u\|^{\frac{\alpha'}{\alpha'+\beta'}}_{L^\infty(\mathbb{B}_{r_1})}\|u\|^{\frac{\beta'}{\alpha'+\beta'}}_{L^\infty(\mathbb{B}_{2r_3})}.$$

By a rescaling argument, we arrive at the conclusion of the lemma. □

We begin to prove Theorem 2. The idea is similar to the proof of Theorem 1. We also use the propagation of smallness argument.

Proof of Theorem 2. We choose a small r such that

$$\sup_{\mathbb{B}_{\frac{r}{2}(0)}} |u| = \epsilon,$$

where $\epsilon > 0$. Since $\sup_{|x|\leq 1}|u(x)| \geq 1$, there should exist some $\bar{x} \in \mathbb{B}_1$ such that $u(\bar{x}) = \sup_{|x|\leq 1}|u(x)| \geq 1$. We select a sequence of balls with radius r centered at $x_0 = 0$, $x_1, \cdots, x_d$ so that $x_{i+1} \in \mathbb{B}_{\frac{r}{2}}(x_i)$ and $\bar{x} \in \mathbb{B}_r(x_d)$, where d depends on the radius r which is to be fixed. Employing the L^∞-version of three-ball lemma (i.e. Lemma 7) with $r_1 = \frac{r}{2}$, $r_2 = r$, and $r_3 = 6r$ and the boundedness assumption of u, we get

$$\|u\|_{L^\infty(\mathbb{B}_r)} \leq C_1 r^{-\frac{n}{2}}\epsilon^\theta \exp{(CM)}$$

where $1 < \theta = \frac{\log 9/8}{\log 9/8 + C\log 16/3} < 1$ and C_1 depends on the L^∞ norm of u, n and m.

Iterating the above argument with L^∞-version of three-ball lemma for ball centered at x_i and using the fact that $\|u\|_{L^\infty(\mathbb{B}_{\frac{r}{2}}(x_{i+1}))} \leq \|u\|_{L^\infty(\mathbb{B}_r(x_i))}$, we have

$$\|u\|_{L^\infty(\mathbb{B}_r(x_i))} \leq C_i\epsilon^{D_i}r^{-\frac{F_i n}{2}}\exp{(E_i M)}$$

for $i = 0, 1, \cdots, d$, where C_i is a constant depending on d and L^∞-norm of u, and D_i, E_i, F_i are constants depending on d. By the fact that $u(\bar{x}) \geq 1$ and $\bar{x} \in \mathbb{B}_r(x_d)$, we obtain

$$\sup_{\mathbb{B}_{\frac{r}{2}(0)}} |u| = \epsilon \geq K_1\exp{(-K_2M)}r^{\frac{K_3 n}{2}},$$

where K_1 is a constant depending on d and L^∞-norm of u, and K_2, K_3 are constants depending on d.

Applying Lemma 7 again centered at origin with $r_2 = \frac{r}{2} > r_1$ and $r_3 = 3r$, where r_1 is sufficiently small, we have

$$K_1\exp{(-K_2M)}r^{\frac{K_3 n}{2}} \leq C\exp{(CM)}r^{-\frac{n}{2}}\|u\|^{\frac{\alpha_3}{\alpha_3+\beta_3}}_{L^\infty(\mathbb{B}_{r_1})}C_0^{\frac{\beta_3}{\alpha_3+\beta_3}}.$$

Recall that C_0 is the L^∞ norm for u in $\mathbb{B}_{10}$. Then

$$K_4^{1+q}\exp{(-(1+q)K_5M)}r^{K_6(1+q)} \leq \|u\|_{L^\infty(\mathbb{B}_{r_1})}, \tag{3.27}$$

where K_4 depends on d and L^∞ norm of u, and K_5, K_6 depend on d,

$$q = \frac{\beta_3}{\alpha_3} = \frac{C\log\frac{4r}{3r_1}}{\log\frac{9}{8}} = -K_7 + C\log\frac{1}{r_1},$$

with constant $K_7 > 0$ depending on r. At this moment we fix the small value of r. For instance, let $r = \frac{1}{100}$. Then the value of d is determined too. The inequality (3.27) implies that

$$\|u\|_{L^\infty(\mathbb{B}_{r_1})} \geq K_8 r_1^{K_9 M},$$

where the constants K_8, K_9 depend on the dimension n, m, and C_0. The proof of Theorem 2 is arrived. $\square$

Thanks to Theorem 2, we are able to prove the following corollary for higher order elliptic equations in (1.8), which characterizes the asymptotic behavior of u at infinity.

Proof of Corollary 1. We adapt the proof in [**K**]. Since u is continuous, we can find $|x_0| = R$ so that $M(R) = \sup_{B_1(x_0)} |u(x)|$. Let

$$u_R(x) = u(R(x + \frac{x_0}{R})) \quad \text{and} \quad \overline{V}_R = \overline{V}(R(x + \frac{x_0}{R})).$$

Then

$$(-\triangle)^m u_R = R^{2m} \overline{V}_R u_R,$$

with $\|u_R\|_{L^\infty} \leq C_0$ and $\|R^{2m}\overline{V}_R\|_{L^\infty} \leq R^{2m}$. So $M = R^{2m}$ in the notation of Theorem 2. If $\bar{x}_0 = \frac{-x_0}{R}$, then $|\bar{x}_0| = 1$ and $u_R(\bar{x}_0) = u(0) = 1$. Hence $\|u_R\|_{L^\infty(B_1)} \geq 1$. Note that $\sup_{\mathbb{B}_1(x_0)} |u(x)| = \sup_{\mathbb{B}_{r_0}} |u_R(y)|$, where $r_0 = \frac{1}{R}$. The conclusion in Theorem 2 leads to

$$\begin{aligned} M(R) = \sup_{\mathbb{B}_{r_0}} |u_R(y)| &\geq Cr_0^{CM} \\ &= C(\tfrac{1}{R})^{CR^{2m}} \\ &= C\exp(-CR^{2m}\log R), \end{aligned}$$

where C depends on n, m and C_0. Thus, the corollary follows. $\square$

4. Strong unique continuation

In the rest of the paper, we will show the strong unique continuation result for higher order elliptic equations by the monotonicity of frequency function. This variant of frequency function is also powerful in obtaining unique continuation results. For strong unique continuation results of semilinear equations and system of equations using frequency function, we refer to [**GL**], [**GL1**] and [**AM**] for the Lamé system of elasticity. Let u be the solution in (1.9). Since we do not need to control the vanishing order of solutions, we assume $\alpha = 0$ for $H(r)$, i.e.

$$H(r) = \sum_{i=1}^{m} \int_{\mathbb{B}_r} u_i^2 \, dx.$$

We can check that

$$H'(r) = \frac{n}{r} H(r) + \frac{1}{r} I(r) \tag{4.1}$$

where

$$I(r) = 2 \sum_{i=1}^{m} \int (x \cdot \nabla u_i) u_i \, dx.$$

We consider the following frequency function

$$N(r) = \frac{I(r)}{H(r)}. \tag{4.2}$$

Similar arguments as the proof of Lemma 4 lead to following monotonicity.

Lemma 8. *There exists a constant C depending only on n, m such that*

$$\exp{(Cr)}(N(r) + (\|\overline{V}\|_{L^\infty_{loc}} + 1)^2)$$

is nondecreasing function of $r \in (0, 1)$.

Based on the monotonicity property in above lemma, we are able to show the proof of Theorem 3.

Proof of Theorem 3. By the equality (4.1), we get

$$(\log \frac{H(r)}{r^n})' = \frac{\overline{N}(r)\exp{(-Cr)} - v^2}{r}, \tag{4.3}$$

where

$$\overline{N}(r) = \exp{(Cr)}(N(r) + v^2).$$

Recall that $v = \|\overline{V}\|_{L^\infty_{loc}} + 1$. Integrating from R to $4R$ for the equality (4.3) yields that

$$\log 4^{-n} \frac{H(4R)}{H(R)} \leq C\overline{N}(1) \log 4,$$

where R is chosen to be small and C depends on n and m. Taking exponential of both sides,

$$H(4R) \leq \exp{(C(\overline{N}(1) + 1))} H(R),$$

that is,

$$\sum_{i=1}^m \int_{\mathbb{B}_{4R}} u_i^2\, dx \leq C \sum_{i=1}^m \int_{\mathbb{B}_R} u_i^2\, dx, \tag{4.4}$$

where C depends on $\overline{N}(1)$. From the decomposition in (3.1) and scaling arguments in (3.24), we have

$$(\sum_{i=1}^m \int_{\mathbb{B}_R} u_i^2\, dx)^{\frac{1}{2}} \leq CR^{-2m}(\|\overline{V}\|_{L^\infty_{loc}} + 1)\|u\|_{L^2(\mathbb{B}_{2R})},$$

Therefore, with the aid of (4.4),

$$\int_{\mathbb{B}_{4R}} u^2\, dx \leq \sum_{i=1}^m \int_{\mathbb{B}_{4R}} u_i^2\, dx \leq C\sum_{i=1}^m \int_{\mathbb{B}_R} u_i^2\, dx \leq CR^{-4m} \int_{\mathbb{B}_{2R}} u^2\, dx.$$

Thus, we get a doubling type estimate

$$\int_{\mathbb{B}_{2R}} |u|^2\, dx \leq CR^{-4m} \int_{\mathbb{B}_R} |u|^2\, dx, \tag{4.5}$$

where C depends on $\overline{N}(1)$, $\|\overline{V}\|_{L^\infty_{loc}}$, n and m. Now we fix R and prove that $u(x) \equiv 0$ on $\mathbb{B}_R$ from (4.5). The argument is standard. See e.g. [**GL**] on page 256-257.

$$\int_{\mathbb{B}_R} u^2\, dx \leq (CR^{-4m})^k \int_{\mathbb{B}_{2^{-k}R}} u^2\, dx = (CR^{-4m})^k |\mathbb{B}_{2^{-k}R}|^\beta \frac{1}{|\mathbb{B}_{2^{-k}R}|^\beta} \int_{\mathbb{B}_{2^{-k}R}} u^2\, dx, \tag{4.6}$$

where the constant β to be fixed. We choose β such that $CR^{-4m}2^{-n\beta} = 1$. It yields that

$$\int_{\mathbb{B}_R} u^2\, dx \leq CR^{n\beta} \frac{1}{|\mathbb{B}_{2^{-k}R}|^\beta} \int_{\mathbb{B}_{2^{-k}R}} u^2\, dx \to 0 \quad \text{as } k \to \infty \tag{4.7}$$

because of (1.10). Then $u \equiv 0$ in $\mathbb{B}_R$. Since we can choose $\mathbb{B}_R$ arbitrarily in Ω, the proof of Theorem 3 follows.

□

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