Radial Symmetry and Regularity of Solutions for Poly-harmonic Dirichlet Problems

Wenxiong Chen [*] Jiuyi Zhu [†]

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Abstract

Let $\mathbf{B} = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and $r = |x|$. We study the poly-harmonic Dirichlet problem

$$\begin{cases} (-\triangle)^m u = f(u) & \text{in } \mathbf{B}, \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B}. \end{cases}$$

Using the corresponding integral equation and the method of moving planes in integral forms, we show that the positive solutions are radially symmetric and monotone decreasing about the origin. We also obtain regularity for solutions.

1 Introduction

In 1979, Gidas-Ni-Nirenberg [GNN] considered the following semilinear elliptic equation

$$\begin{cases} -\triangle u = f(u) & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{1}$$

where $\mathbf{B} = B_1(0)$ is the unit ball in $\mathbb{R}^n$. Under the condition that $f(\cdot)$ is locally Lipschitz continuous, by using the method of moving planes, they proved that every positive smooth solution is radially symmetric and monotone decreasing about the origin.

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In this paper, we study a more general problem, the poly-harmonic operator with Dirichlet boundary conditions

$$\left\{\begin{array}{ll} (-\triangle)^m u = f(u) & \text{in } \mathbf{B}, \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B}. \end{array}\right. \tag{2}$$

where $r = |x|$, m is any positive integer.

As usual, we say that u is a weak solutions of (2) in Sobolev space $H_0^m(\mathbf{B})$, if it satisfies

$$< u, v >_m = \int_{\mathbf{B}} f(u(x))v(x)dx, \;\; \forall\, v \in H_0^m(\mathbf{B}), \tag{3}$$

where

$$< u, v >_m = \left\{\begin{array}{ll} \int_{\mathbf{B}} \triangle^{\frac{m}{2}} u \cdot \triangle^{\frac{m}{2}} v \, dx, & m \,\text{even}, \\ \int_{\mathbf{B}} \left(\bigtriangledown \triangle^{\frac{(m-1)}{2}} u\right) \cdot \left(\bigtriangledown \triangle^{\frac{(m-1)}{2}} v\right) \, dx, & m \,\text{odd} \end{array}\right.$$

is an inner product in $H_0^m(\mathbf{B})$.

If u is a solution of (2), from the standard regularity theorem in [ADN], then $u \in H_0^{2m}(\mathbf{B})$. Multiplying both sides of (2) by the Green's function $G(x, y)$ of $(-\triangle)^m$ in $\mathbf{B}$ with the Dirichlet boundary conditions, then after integration by parts, we arrive at the integral equation:

$$u(x) = \int_{\mathbf{B}} G(x, y) f(u(y)) \, dy. \tag{4}$$

We assume $f(u(x)) \in L^2(B)$ satisfies the following conditions:

(f_1) $f : [0, \;\infty) \to R$ is nondecreasing, $f(0) \geq 0$, and either one of the following

(f_2) $f'(\cdot)$ is monotonic,

$$f'(u) \in \left\{\begin{array}{ll} L^{\frac{n}{2m}}(\mathbf{B}), & \text{if } n > 2m, \\ L^s(\mathbf{B}), \;\text{for some } s > 1, & \text{if } n \leq 2m, \end{array}\right.$$

or

$(\tilde{f}_2)$

$$|f'(u)| \leq C_1 |u|^{\beta_1} + C_2 |u|^{\beta_2} + C_3,$$

where C_1, C_2, C_3 are nonnegative constants, β_1 is some nonnegative constant, and β_2 is some non-positive constant. If $C_1 > 0$, we require $|u|^{\beta_1} \in L^{\frac{n}{2m}}(\mathbf{B})$, and if $C_2 > 0$, we need $|u|^{\beta_2} \in L^{\frac{n}{2m}}(\mathbf{B})$. Here $f'(\cdot)$ is in the sense of distribution. In the case $2m \geq n$, we have no restriction on β_1 due to Sobolev imbedding, however, we need $|u|^{\beta_2} \in L^s(\mathbf{B})$ for some $s > 1$.

We will use the method of moving plane in integral forms on integral equation (4) and prove

Theorem 1. *Assume that $f(\cdot)$ satisfies condition (f_1) and either (f_2) or ($\tilde{f_2}$), then every positive solutions of (2) is radially symmetric about the origin and strictly decreasing in the radial direction.*

Remark 1.1. *(1) Obviously, in the special case when $m = 1$, (2) is reduced to (1) considered in the elegant paper [GNN]. If $f(u)$ is locally Lipschitz continuous, then it satisfies our condition ($\tilde{f_2}$) with $C_1 = C_2 = 0$. However, there are functions, for example, $f(u) = u^\alpha$ $(0 < \alpha < 1)$, that satisfy ($\tilde{f_2}$), but are not locally Lipschitz continuous even if $u(x)$ is differentiable. Our Theorem seems to expand the results of [GNN] in this respect.*

(2) A similar method in proving our theorem can also be applied to the case when $f(u) = u^s + u^t$ with $0 \leq s < 1$ and $t > 1$, although this kind of $f(\cdot)$ is not Locally Lipschitz.

(3) We would like to mention that similar results have also been established by Berchio, Gazzola, and Weth in [BGW]. In [BGW], they required $u \in L^\infty(\mathbf{B}) \bigcap H_0^m(\mathbf{B})$, f satisfies (f_1) and be continuous. Here we only need $u \in H_0^m(\mathbf{B})$. When $f(u) = u^p$, our conditions seem weaker. Our results and theirs complement each other due to different approaches.

For more general boundary value problems for higher order elliptic equations, please see [ADN].

(4) For $m \geq 2$, the sign assumptions on f seem to be necessary in order to attain the radial monotonicity of u, as indicated by the counterexample in [S].

(5) Our method can also be applied to more general function $f(|x|, u)$, if $f(|x|, u)$ is non-increasing with respect to $|x|$ and with proper growth.

The method of moving planes was invented by Soviet mathematician Alexandrov in the 1950s, then it is further developed by Serrin, Gidas, Ni, Nirenberg, Caffarelli, Spruck, and many others. It is mainly based on various maximum principles for partial differential equations. Recently, Chen, Li, and Ou [CLO] introduced a new approach–the method of moving planes in integral forms–to obtain symmetry, monotonicity, and nonexistence of solutions for integral equations. It is entirely different from the traditional methods of moving planes used for partial differential equations. Instead of relying on the differentiability and maximum principles of the structure, a global integral norms are estimated. In many cases, one can prove that a PDE system is equivalent to an integral system (see [CLO] [CL2] [CL3] [CL5]). Hence, the method of moving planes in integral forms can also be adapted to obtain symmetry for solutions of PDEs.

Previously, the method of moving planes in integral forms were applied to equations in the whole $\mathbb{R}^n$, and it is the first time in this paper we adapt it to a bounded domain with boundary conditions. As one will see in the proof, there are some difficulties needed to be overcome and thus some new approaches are involved.

For more articles concerning the method of moving planes on integral equations, please see [CJLL] [CL] [CL1] [CL2] [CL3] [CL7] [CLO1] [Ha] [HWY] [JL] [LiY4] [LL] [LM] [LQ] [LZ] [MC] [MC1] [MC2] [MZ] and the references therein.

Besides symmetry, we also establish regularity for the solutions.

Theorem 2. *Let $u(x)$ be a positive solution of (4). Assume that $u(x) \in L^q(\mathbf{B})$ for some $q > \frac{n}{n-2m}$, and*

$$\int_{\mathbf{B}} |f(u(y))|^{n/2m} dy < \infty. \tag{5}$$

Then u is uniformly bounded in $\mathbf{B}$.

The paper is arranged as follows. In Section 1, we present some properties of the Green's function for the poly-harmonic Dirichlet problem in the ball. In Section 2, we prove Theorem **1** using the method of moving planes in integral forms. In Section 3, we derive Theorem **2** by using a regularity lifting method. In this paper, we use C to denote various positive constants whose value may vary from line to line.

2 Properties of Green's Functions

In this section, we introduce some properties of the Green's function $G(x, y)$ of $(-\triangle)^m$on the unit ball $\mathbf{B}$ with Dirichlet boundary conditions.

For each fixed $y \in \mathbf{B}$, the Green's function is the solution of

$$\left\{ \begin{array}{ll} (-\triangle)^m G(x,y) = \delta(x-y) & \text{in } \mathbf{B}, \\ G = \dfrac{\partial G}{\partial r} = \cdots = \dfrac{\partial^{m-1} G}{\partial r^{m-1}} = 0 & \text{on } \partial \mathbf{B}. \end{array} \right. \tag{6}$$

Thanks to Boggio [B], it can be expressed explicitly in terms of x and y. To this end, define, for $x, y \in \mathbb{R}^n$,

$$d(x,y) = |x-y|^2$$

and

$$\theta(x,y) = \left\{ \begin{array}{ll} (1-|x|^2)(1-|y|^2) & \text{if } \ x, y \in \mathbf{B}, \\ 0 & x \notin \mathbf{B} \ or \ y \notin \mathbf{B}. \end{array} \right. \tag{7}$$

Then for $x, y \in \mathbf{B}$, $x \neq y$, we have the following representation

$$\begin{aligned} G(x,y) &= C_n^m |x-y|^{2m-n} \int_0^{\frac{\theta(x,y)}{|x-y|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} \, dz \\ &= C_n^m H(d(x,y), \theta(x,y)). \end{aligned}$$

Here C_n^m is a positive constant and

$$H : (0, \ \infty) \times [0, \ \infty) \to R, \qquad H(s,t) = s^{m-\frac{n}{2}} \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} \, dz.$$

For $\lambda \in (-1, \ 0)$, let

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) \in \mathbf{B} \mid x_1 < \lambda\}$$

and

$$\Sigma_\lambda^C = \mathbf{B} \setminus \Sigma_\lambda,$$

the complement of Σ_λ in $\mathbf{B}$.

The following lemma states some properties of the Green's function, which will be used in the next section. The first part was established in [BGW]. Here we present a simpler proof.

Lemma 2.1. *Let* $\lambda \in (-1, 0)$.

(i) For any

$$x, y \in \Sigma_\lambda, \quad x \neq y,$$

we have

$$G(x_\lambda, y_\lambda) > max\{G(x_\lambda, y), G(x, y_\lambda)\} \tag{8}$$

and

$$G(x_\lambda, y_\lambda) - G(x, y) > |G(x_\lambda, y) - G(x, y_\lambda)|. \tag{9}$$

(ii) For any

$$x \in \Sigma_\lambda,\ y \in \Sigma_\lambda^C,$$

it holds

$$G(x_\lambda, y) > G(x, y). \tag{10}$$

Proof.

Since $x, y \in \Sigma_\lambda$, it is easy to verify that

$$d(x_\lambda, y_\lambda) < d(x, y_\lambda) \text{ and } \theta(x_\lambda, y_\lambda) > \theta(x, y_\lambda). \tag{11}$$

Moreover we have

$$\begin{aligned} \theta(x_\lambda, y_\lambda) &> \max(\theta(x, y_\lambda), \theta(x_\lambda, y)) \\ &\geq \min(\theta(x, y_\lambda), \theta(x_\lambda, y)) \\ &> \theta(x, y). \end{aligned} \tag{12}$$

Consider

$$\begin{aligned} G(x, y) = C_n^m H(s, t) &= C_n^m s^{m-\frac{n}{2}} \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} dz \\ &= C_n^m \int_0^t \frac{z^{m-1}}{(z+s)^{\frac{n}{2}}} \end{aligned}$$

with

$$t = \theta(x, y) \text{ and } s = d(x, y).$$

Then for $s, t > 0$,

$$\frac{\partial H}{\partial s} = -\frac{n}{2} \int_0^t \frac{z^{m-1}}{(z+s)^{\frac{n}{2}+1}} < 0, \tag{13}$$

$$\frac{\partial H}{\partial t} = \frac{t^{m-1}}{(t+s)^{\frac{n}{2}}} > 0 \tag{14}$$

and

$$\frac{\partial^2 H}{\partial t \partial s} = -\frac{n}{2} \frac{t^{m-1}}{(t+s)^{\frac{n}{2}+1}} < 0. \tag{15}$$

(i) From (11), (13), and (14), we arrive at (8).

While by (12) and (15), we have

$$\begin{aligned}
G(x_\lambda, y_\lambda) - G(x,y) &= C_n^m \int_{\theta(x,y)}^{\theta(x_\lambda,y_\lambda)} \frac{\partial H(d(x,y),t)}{\partial t}\,dt \\
&> C_n^m \int_{\theta(x,y)}^{\theta(x_\lambda,y_\lambda)} \frac{\partial H(d(x_\lambda,y),t)}{\partial t}\,dt \\
&\geq C_n^m \int_{\theta(x,y_\lambda)}^{\theta(x_\lambda,y)} \frac{\partial H(d(x_\lambda,y),t)}{\partial t}\,dt \\
&= C_n^m\, |H(d(x_\lambda,y),\,\theta(x_\lambda,y)) - H(d(x,y_\lambda),\,\theta(x,y_\lambda))| \\
&= |G(x_\lambda,y) - G(x,y_\lambda)|.
\end{aligned}$$

Here we have used the fact that $d(x_\lambda, y) = d(x, y_\lambda)$.

(ii) Noticing that for $x \in \Sigma_\lambda$ and $y \in \Sigma_\lambda^C$, we have

$$|x_\lambda - y| < |x - y| \quad \text{and} \quad 1 - |x|^2 < 1 - |x_\lambda|^2.$$

Then (10) follows immediately from (13) and (14).

This completes the proof of Lemma **2.1**.

3 Symmetry of Solutions

In this section, we prove Theorem **1**.

Let $\lambda \in (-1, 0)$,

$$\Sigma_\lambda = \{x \in \mathbf{B} \mid x_1 < \lambda\},$$

$$T_\lambda = \{x \in \mathbb{R}^n | \, x_1 = \lambda\},$$

$$x_\lambda = \{2\lambda - x_1,\, x_2, \cdots, x_n\}$$

be the reflection of the point x about the hyperplane T_λ, and

$$\tilde{\Sigma_\lambda} = \{x_\lambda \mid x \in \Sigma_\lambda\},$$

the image of Σ_λ about the plane T_λ. Set

$$u_\lambda(x) = u(x_\lambda).$$

To prove Theorem **1**, we compare the value of $u(x)$ with $u_\lambda(x)$ in Σ_λ. The proof consists of two steps. In Step 1, we show that for λ sufficiently close to -1, we have

$$w_\lambda(x) := u_\lambda(x) - u(x) \geq 0 \quad a.e. \tag{16}$$

This provides a starting point for us to move the plane T_λ along the x_1 direction. In Step 2, we move the plane continuously to the right as long as inequality (16) holds. We show that the plane can be moved all the way to $\lambda = 0$ and thus derive

$$u(-x_1, x_2, \cdots, x_n) \leq u(x_1, x_2, \cdots, x_n), \ \ \forall\, x \in \mathbf{B}, x_1 \geq 0. \tag{17}$$

Similarly, we can start the plane T_λ near $\lambda = 1$ and move it to the left to the limiting position T_0 to deduce

$$u(-x_1, x_2, \cdots, x_n) \geq u(x_1, x_2, \cdots, x_n), \ \ \forall\, x \in \mathbf{B}, x_1 \geq 0. \tag{18}$$

Combining inequalities (17) and (18), we conclude that $u(x)$ is symmetric about the plane T_0. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ is radially symmetric and decreasing about the origin.

The following lemmas are key ingredients in our integral estimates.

Lemma 3.1. *For any $x \in \Sigma_\lambda$, it holds*

$$u(x) - u(x_\lambda) \leq \int_{\Sigma_\lambda} [G(x_\lambda,\, y_\lambda) - G(x,\, y_\lambda)][f(u(y)) - f(u_\lambda(y))].$$

Proof. Obviously, we have

$$\begin{aligned} u(x) \ &= \ \int_{\Sigma_\lambda} G(x\,, y) f(u(y))\, dy + \int_{\Sigma_\lambda} G(x\,, y_\lambda) f(u_\lambda(y))\, dy \\ &+ \ \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma_\lambda}} G(x, y) f(u(y)) dy \end{aligned}$$

and

$$\begin{aligned} u_\lambda(x) \ &= \ \int_{\Sigma_\lambda} G(x_\lambda\,, y) f(u(y))\, dy + \int_{\Sigma_\lambda} G(x_\lambda\,, y_\lambda) f(u_\lambda(y))\, dy \\ &+ \ \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma_\lambda}} G(x_\lambda, y) f(u(y)) dy. \end{aligned}$$

Now by properties (9) and (10) of the Green's function and the non-negativeness assumption on f, we arrive at

$$\begin{aligned} u(x)-u(x_\lambda) &\leq \int_{\Sigma_\lambda}[G(x_\lambda, y_\lambda)-G(x, y_\lambda)][f(u(y))-f(u_\lambda(y))]\,dy \\ &+ \int_{\Sigma_\lambda^C\setminus\tilde{\Sigma_\lambda}}[G(x,y)-G(x_\lambda,y)]f(u(y))dy \\ &\leq \int_{\Sigma_\lambda}[G(x_\lambda, y_\lambda)-G(x, y_\lambda)][f(u(y))-f(u_\lambda(y))]\,dy. \end{aligned}$$

This completes the proof of the lemma.

Lemma 3.2. *(*An equivalent form of the Hardy-Littlewood-Sobolev inequality*)*

Assume $0<\alpha<n$ *and* $\Omega\subset\mathbb{R}^n$. *Let* $g\in L^{\frac{np}{n+\alpha p}}(\Omega)$ *for* $\frac{n}{n-\alpha}<p<\infty$. *Define*

$$Tg(x)=\int_\Omega \frac{1}{|x-y|^{n-\alpha}}g(y)dy.$$

Then

$$\|Tg\|_{L^p(\Omega)}\leq C(n,p,\alpha)\|g\|_{L^{\frac{np}{n+\alpha p}}(\Omega)}. \tag{19}$$

The proof of this Lemma is standard and can be found in book [CL4].

Proof of Theorem 1.

Step1. Define

$$\Sigma_\lambda^-=\{x\in\Sigma_\lambda | u(x)>u_\lambda(x)\},$$

the set where inequality (16) is violated. We are going to show that Σ_λ^- is almost empty by estimating a certain integral norm on it.

By virtue of Lemma **3.1**, property (8) of the Green's function, and the monotonicity of $f(\cdot)$, we have, for any $x\in\Sigma_\lambda^-$,

$$\begin{aligned} 0 &< u(x)-u_\lambda(x) \\ &\leq \int_{\Sigma_\lambda^-}[G(x_\lambda, y_\lambda)-G(x,y_\lambda)][f(u(y))-f(u_\lambda(y))]dy \\ &\leq \int_{\Sigma_\lambda^-} G(x_\lambda, y_\lambda)[f(u(y))-f(u_\lambda(y))]\,dy \\ &\leq \int_{\Sigma_\lambda^-} G(x_\lambda,y_\lambda)|f'(\xi(y))||w_\lambda(y)|dy, \end{aligned} \tag{20}$$

where $\xi(y)$ is valued between $u(y)$ and $u_\lambda(y)$ by Mean Value Theorem.

Recall the representation formula:

$$G(x,y) = C_n^m |x-y|^{2m-n} \int_0^{\frac{\theta(x,y)}{|x-y|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} dz. \tag{21}$$

We consider three possible cases.

Case (i): $2m < n$. By (21), we have

$$G(x_\lambda, y_\lambda) \leq \frac{C}{|x_\lambda - y_\lambda|^{n-2m}} = \frac{C}{|x-y|^{n-2m}}.$$

It follows from (20) that, for any $x \in \Sigma_\lambda^-$,

$$\begin{aligned} 0 &< u(x) - u_\lambda(x) \\ &\leq C \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-2m}} |f'(\xi(y))| |w_\lambda(y)| dy. \end{aligned} \tag{22}$$

Applying the *Hardy-Littlewood-Sobolev* inequality and the *Hölder* inequality, for any $q > \frac{n}{n-2m}$, (in particular, when $q = \frac{2n}{n-2m}$, $w_\lambda \in L^q(\mathbf{B})$ by Sobolev imbedding), we have

$$\begin{aligned} \|w_\lambda(x)\|_{L^q(\Sigma_\lambda^-)} &\leq C \|f'(\xi(x)) w_\lambda(x)\|_{L^{\frac{nq}{n+2mq}}(\Sigma_\lambda^-)} \\ &\leq C \|f'(\xi(x))\|_{L^{\frac{n}{2m}}(\Sigma_\lambda^-)} \|w_\lambda(x)\|_{L^q(\Sigma_\lambda^-)}. \end{aligned} \tag{23}$$

From assumption (f_2), for λ sufficiently close to -1, we have

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq \frac{1}{2} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \tag{24}$$

This implies that $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, therefore Σ_λ^- must be measure zero.

Case (ii): $2m = n$. By (21), for any $a > 0$, it holds

$$G(x,y) \leq \int_0^{\frac{\theta(x,y)}{|x-y|^2}} \frac{1}{1+z} dz \leq \ln\left(1 + \frac{C}{|x-y|^2}\right) \leq \frac{C_a}{|x-y|^a}.$$

Using (20) and *Young's* inequality with $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$; $p, q, r > 1$; and $q > r$, we derive

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C_a \|\frac{1}{|x|^a}\|_{L^p(\Sigma_\lambda^-)} \|f'(\xi(x)) w_\lambda(x)\|_{L^r(\Sigma_\lambda^-)}.$$

Consequently, by *Hölder* inequality,

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C_a \|\frac{1}{|x|^a}\|_{L^p(\Sigma_\lambda^-)} \|f'(\xi(x))\|_{L^{\frac{p}{p-1}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}.$$

Choose p and a, such that $\frac{p}{p-1} = s$ (as given in (f_2)), and $ap < n$. Then for λ sufficiently close to -1, (24) holds.

Case (iii): $2m > n$. Again by (21), we have

$$G(x,y) \leq C|x-y|^{2m-n}\left(1 + \frac{\theta(x,y)}{|x-y|^2}\right)^{m-\frac{n}{2}} \leq C_1.$$

Then it follows from (20) that

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C[\mu(\Sigma_\lambda^-)]^{\frac{1}{q}} \|f'(\xi(x))\|_{L^{\frac{q}{q-1}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}.$$

Noticing that as λ sufficiently close to -1, $\mu(\Sigma_\lambda^-)$ is very small, and by (f_2), we again arrive at (24). Here we have chosen $\frac{q}{q-1} = s$ (as given in (f_2)), this is possible because $w_\lambda \in L^q(\mathbf{B})$ for any $q > 1$ by Sobolev imbedding.

Therefore, in all three cases, for λ close to -1, inequality (16) holds.

Step 2. We now move the plane $x_1 = \lambda$ continuously toward the right as long as inequality (16) holds to its limiting position. Define

$$\lambda_0 = \sup\{\lambda \in (-1\,,0) | w_\mu(x) \geq 0,\ x \in \Sigma_\mu, \mu \leq \lambda\}. \tag{25}$$

We argue that λ_0 must be 0.

Otherwise, suppose $\lambda_0 < 0$, we must have

$$u_{\lambda_0}(x) > u(x),\ \forall x \in \Sigma_{\lambda_0}.$$

In fact, similar to the proof of Lemma **3.1**, we have

$$\begin{aligned} u_\lambda(x) - u(x) \ \geq\ & \int_{\Sigma_\lambda} [G(x_\lambda,\, y_\lambda) - G(x,\, y_\lambda)][f(u_\lambda(y)) - f(u(y)]\, dy \\ & + \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma_\lambda}} [G(x_\lambda, y) - G(x,y)] f(u(y)) dy. \end{aligned}$$

If there exists some point $x_0 \in \Sigma_{\lambda_0}$ such that $u(x_0) = u_{\lambda_0}(x_0)$, then by Lemma **2.1**, the monotonicity and nonnegative-ness of f, we derive

$$f(u_{\lambda_0}(y)) \equiv f(u(y)),\ \forall\, y \in \Sigma_{\lambda_0} \ \text{ and } \ f(u(y)) \equiv 0,\ \forall\, y \in \Sigma_\lambda^C \setminus \tilde{\Sigma_\lambda}. \tag{26}$$

On the other hand,

$$\begin{aligned}
& u(x) - u_{\lambda_0}(x) \\
= & \int_{\Sigma_{\lambda_0}} [G(x,y) - G(x_{\lambda_0}, y)] f(u(y))\, dy + \int_{\Sigma_{\lambda_0}} [G(x, y_{\lambda_0}) - G(x_{\lambda_0}, y_{\lambda_0})] f(u_{\lambda_0}(y))\, dy \\
+ & \int_{\Sigma_{\lambda_0}^C \setminus \tilde{\Sigma_\lambda}} [G(x_{\lambda_0}, y) - G(x,y)] f(u(y)) dy.
\end{aligned}$$

Combining this with (26) and noticing that (from Lemma **2.1**)

$$G(x,y) - G(x_{\lambda_0}, y) + G(x, y_{\lambda_0}) - G(x_{\lambda_0}, y_{\lambda_0}) < 0,$$

we deduce

$$f(u(y)) \equiv f(u_{\lambda_0}(y)) \equiv 0, \ \forall\, y \in \Sigma_{\lambda_0}.$$

Consequently

$$f(u(y)) \equiv 0, \ \forall\, y \in \mathbf{B}.$$

This implies $u \equiv 0$ by the uniqueness of the Dirichlet problem, which is a contradiction with our assumption that $u > 0$. Therefore we must have

$$u_{\lambda_0}(x) > u(x), \ \ \forall\, x \in \Sigma_{\lambda_0}. \tag{27}$$

By virtue of *Lusin* theorem, for any $\delta > 0$, there exists a closed subset F_δ of Σ_{λ_0}, with $\mu(\Sigma_{\lambda_0} \setminus F_\delta) < \delta$, such that $w_{\lambda_0}|_{F_\delta}$ is continuous (with respect to x), and hence $w_\lambda|_{F_\delta}$ is continuous with respect to λ for λ close to λ_0. By (27), there exists $\epsilon > 0$ such that for all $\lambda \in [\lambda_0, \ , \lambda_0 + \epsilon)$, it holds

$$w_\lambda(x) \geq 0, \ \forall\, x \in F_\delta.$$

It follows that, for such λ,

$$\mu(\Sigma_\lambda^-) \leq \mu(\Sigma_{\lambda_0} \setminus F_\delta) + \mu(\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \leq \delta + 2\epsilon.$$

As we did in Step 1, in the case $2m < n$, choose δ and ϵ sufficiently small so that

$$C \| f'(\xi(x)) \|_{L^{\frac{n}{2m}}(\Sigma_\lambda^-)} \leq \frac{1}{2}.$$

Consequently from (23), we have $\|w_\lambda(x)\|_{L^q(\Sigma_\lambda^-)} = 0$, and hence Σ_λ^- must be measure zero, and hence

$$w_\lambda(x) \geq 0, \ a.e. \ x \in \Sigma_\lambda, \ \lambda \in [\lambda_0, \lambda_0 + \epsilon).$$

This contradicts with the definition of λ_0. Therefore, we must have $\lambda_0 = 0$. We now have completed the proof of the theorem in the case $2m < n$. It is similar for other cases.

4 Regularity of Solutions

In this section, we prove regularity of positive solutions $u(x)$ for the polyharmonic Dirichlet problems, and the following lemma from [CL4] is a key ingredient in our proof.

Lemma 4.1. *(Regularity Lifting) Let V be a Hausdorff topological vector space. Suppose there are two extended norms (i.e. the norm of an element in V might be infinity) defined on V,*

$$\|\cdot\|_X,\ \|\cdot\|_Y : V \to [0, \infty].$$

Assume that the spaces

$$X := \{v \in V : \|v\|_X < \infty\} \ \text{ and } \ Y := \{v \in V : \|v\|_Y < \infty\}$$

are complete under the corresponding norms, and the convergence in X or in Y implies the convergence in V.

Let T be a contracting map from X into itself and from Y into itself. Assume that $f \in X$, and that there exits a function $g \in Z := X \cap Y$ such that $f = Tf + g$ in X. Then f also belongs to Z.

Proof of Theorem 2

We first show that

$$u \in L^p(\mathbf{B}), \quad \text{for any } p > \frac{n}{n-2m}. \tag{28}$$

In the following, we assume $2m < n$. In this case, by the representation of the Green's function, we have

$$G(x, y) \leq \frac{C}{|x-y|^{n-2m}}. \tag{29}$$

For any real number $a > 0$, let $A = \{x \in \mathbf{B} \mid |u(x)| > a\}$ and

$$u_a(x) = \begin{cases} u(x), & \text{if } x \in A, \\ 0, & \text{elsewhere.} \end{cases}$$

Set $u_b(x) = u(x) - u_a(x)$. Then obviously

$$f(u) = f(u_a)\chi_A + f(u_b)\chi_D,$$

where χ_A is the characteristic function on the set A and $D = \mathbf{B} \setminus A$.

Define the linear operator

$$T_a w(x) = \int_{\mathbf{B}} G(x,y) \frac{f(u_a(y))\chi_A(y)}{u_a(y)} w(y)\, dy,$$

and write

$$I(x) = \int_{\mathbf{B}} G(x,y) f(u_b(y))\chi_D(y)\, dy.$$

Then obviously, u satisfies the equation

$$u(x) = T_a u(x) + I(x), \quad \forall\, x \in \mathbf{B}. \tag{30}$$

We prove that, for a sufficiently large, T_a is a contracting map from $L^p(\mathbf{B})$ to $L^p(\mathbf{B})$, for any $p > \frac{n}{n-2m}$. In fact, by (29) and *HLS* inequality,

$$\|T_a w\|_{L^p(\mathbf{B})} \leq C \|\frac{f(u_a)\chi_A}{u_a} w\|_{L^{\frac{np}{n+2mp}}(\mathbf{B})} \leq C\|f(u)w\|_{L^{\frac{np}{n+2mp}}(A)}.$$

Here for simplicity, we may assume $a \geq 1$.

Then by *Hölder* inequality,

$$\|T_a w\|_{L^p(\mathbf{B})} \leq C\|f(u)\|_{L^{\frac{n}{2m}}(A)} \|w\|_{L^p(\mathbf{B})}.$$

Under the integrability assumption on $f(u)$ in Theorem 2, we can choose a sufficiently large, so that the measure of A is small and hence

$$\|T_a w\|_{L^p(\mathbf{B})} \leq \frac{1}{2}\|w\|_{L^p(\mathbf{B})}.$$

Therefore T_a is a contracting operator from $L^p(\mathbf{B})$ to $L^p(\mathbf{B})$.

From the definition of $I(x)$, it is obviously bounded. Now (28) is a consequence of the Regularity Lifting Lemma 4.1.

Finally, by *Hölder* inequality, one can easily see that $u \in L^\infty(\mathbf{B})$. This completes the proof of the Theorem.

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Addresses and e-mails
Wenxiong Chen
College of Mathematics and Information Science
Henan Normal University, P. R. China and
Department of Mathematics
Yeshiva University
New York NY 10033 USA
wchen@yu.edu

Jiuyi Zhu
Department of Mathematics
Wayne state university
Detroit MI 48202 USA
jiuyi.zhu@wayne.edu