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Liouville-type Theorems for Fully Nonlinear Elliptic Equations and Systems in Half Spaces

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Abstract

The main purpose of this paper is to establish Liouville-type theorems and decay estimates for viscosity solutions to a class of fully nonlinear elliptic equations or systems in half spaces without the boundedness assumptions on the solutions. Using the blow-up method and doubling lemma of [18], we remove the boundedness assumption on solutions which was often required in the proof of Liouville-type theorems in the literature. We also prove the Liouville-type theorems for supersolutions of a system of fully nonlinear equations with Pucci extremal operators in half spaces. Liouville theorems and decay estimates for high order elliptic equations and systems have also been established by the authors in an earlier work [15] when no boundedness assumption was given on the solutions.

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1 Introduction

This article is devoted to the study of Liouville-type theorems for nonnegative viscosity solution or supersolutions of a class of fully nonlinear uniformly elliptic equations and systems in a half space $\mathbb{R}^n_+$, i.e. either

$$\begin{cases} F(x, D^2u) + u^p = 0 & \text{in } \mathbb{R}^n_+, \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{1.1}$$

or

$$\begin{cases} F(x, D^2u) + v^p = 0 & \text{in } \mathbb{R}^n_+, \\ F(x, D^2v) + u^q = 0 & \text{in } \mathbb{R}^n_+ \end{cases} \tag{1.2}$$

where $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | x_n > 0\}$ with $n \geq 2$. A continuous function $F : \mathbb{R}^n \times S_n \to \mathbb{R}$ is referred to as an uniformly elliptic equation with ellipticity $0 < \lambda \leq \Lambda$ if for all $M, P \in S_n$ with $P \geq 0$ (nonnegative definite), it holds that

$$\lambda tr(P) \leq F(x, M + P) - F(x, M) \leq \Lambda tr(P), \tag{1.3}$$

where S_n is the space of all real symmetric $n \times n$ matrix, and $tr(P)$ is the trace of $P \in S_n$.

Liouville-type theorems are powerful tools in proving a priori bounds for nonnegative solutions in a bounded domain. They are widely applied in obtaining a priori estimate for solutions of elliptic equations in the literature. Using the "blow-up" method (also called rescaling argument) [12], an equation in a bounded domain will blow up into another equation in the whole Euclidean space or a half space. With the aid of the corresponding Liouville-type theorem in the Euclidean space $\mathbb{R}^n$ and half space $\mathbb{R}^n_+$ and a contradiction argument, the a priori bounds could be readily derived. Moreover, the existence of nonnegative solutions to elliptic equations is established by the topological degree method using a priori estimates (see. e.g. [10]).

In this paper we mainly consider the model in the case when $F(x, D^2u) = \mathcal{M}^+_{\lambda,\Lambda}(D^2u)$. Here $\mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ is the Pucci extremal operator with parameters $0 < \lambda \leq \Lambda$, defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i$$

for any symmetric $n \times n$ matrix M, where $e_i = e_i(M), i = 1, \cdots, n$, denotes the eigenvalue of M. While $\mathcal{M}^-_{\lambda,\Lambda}(M)$ is defined as

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i > 0} e_i.$$

Pucci's operators are extremal in the sense that

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} tr(AM),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} tr(AM)$$

with
$$\mathcal{A}_{\lambda,\Lambda} = \{A \in S_n : \lambda|\xi|^2 \leq A\xi \cdot \xi^T \leq \Lambda|\xi|^2,\ \forall \xi \in \mathbb{R}^n\}.$$
If the operator F is uniformly elliptic with ellipticity constant $0 < \lambda \leq \Lambda$, it results in
$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leq F(x, M) \leq \mathcal{M}^+_{\lambda,\Lambda}(M)$$
when $F(x, O) = 0$. We refer to the monograph [4] for more details on these operators. Notice that $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are not in the divergence form.

When $\lambda = \Lambda = 1$, $\mathcal{M}^\pm_{\lambda,\Lambda}$ coincide with the Laplace operators. Then (1.1) with $F(x, D^2u)$ replaced by $\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u)$ becomes the
$$\triangle u + u^p = 0 \qquad \text{in } \mathbb{R}^n_+. \tag{1.4}$$
It is well known that (1.4) does not have positive supersolutions in the half space for $1 < p < \frac{n+1}{n-1}$, and does not have nonnegative solution for $1 < p < \frac{n+2}{n-2}$ with u vanishing on the boundary.

In view of these results for the semilinear equation (1.4), it would be interesting to understand the structure of solutions for (1.1) and (1.2). Unlike the case of the semilinear equations, the popular technique of Kelvin transform with moving plane method is no longer available. We also note that there is no variational structure for fully nonlinear elliptic equations, even for the Pucci extremal operators. Those impose new difficulties for studying Liouville-type results. In [6], Cutri and Leoni establish the following non-existence results in the spirit of the Hadamard three circle theorem [17]. In particular, they have also shown that the critical exponent
$$p^+ := \frac{\tilde{n}}{\tilde{n} - 2}$$
is optimal for supersolutions in (1.5), where
$$\tilde{n} = \frac{\lambda}{\Lambda}(n - 1) + 1.$$
It exhibits a nontrivial solution for (1.5) if $p > p^+$. Namely, it is stated as the following lemma.

Lemma 1.1 *Assume that $n \geq 3$. If $1 < p \leq p^+$ or ($1 < p < \infty$ if $\tilde{n} \leq 2$), then the only viscosity supersolution of*
$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p = 0 & \text{in } \mathbb{R}^n, \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.5}$$
is $u \equiv 0$.

With the help of moving plane method and the above Liouville-type theorem, Quaas and Sirakov [19] make use of the idea of [8] and obtain a Liouville-type result in a half space. They first prove the solution of (1.6) is non-decreasing in x_n direction, then it leads to the same problem in $\mathbb{R}^{n-1}$ after a limiting process, which allows them to use Lemma 1.1. Under the boundedness assumption, they show that

Lemma 1.2 *Let $n \geq 3$ and $\tilde{p}^+ := \frac{\lambda(n-2)+\Lambda}{\lambda(n-2)-\Lambda}$. Then the equation*

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p = 0 & \text{in } \mathbb{R}^n_+, \\ u \geq 0 & \text{in } \mathbb{R}^n_+, \\ u = 0 & \text{in } \partial\mathbb{R}^n_+ \end{cases} \tag{1.6}$$

has no nontrivial bounded solution, provided $1 < p \leq \tilde{p}^+$ and $\lambda(n-2) > \Lambda$ (or $1 < p < \infty$ if $\lambda(n-2) \leq \Lambda$).

Note that $\tilde{p}^+ > p^+$ for $\lambda(n-2) > \Lambda$. We are interested in the boundedness assumption in Lemma 1.2. As we know, boundedness assumptions are often imposed in deriving such Liouville-type theorem in half spaces. Using the Doubling Lemma recently developed in [18] (see Section 2) and a blow-up technique, we indeed show that the boundedness assumption is unnecessary for such equations. Similar ideas have been applied to derive Liouville type theorems for solutions to higher order elliptic equations and systems in our recent paper [15]. Our strategy is based on a contradiction argument. We suppose that the solution u in (1.6) is unbounded. By the Doubling Lemma and blow-up method, the equation (1.6) will become an equation in a whole Euclidean space or a half space. We will then arrive at a contradiction under a certain range of p, which means that the solution u has to be bounded. Applying Lemma 1.2 again, we obtain the Liouville-type results. In this paper, we first obtain the following result.

Theorem 1.1 *Let $n \geq 3$. For $1 < p \leq p^+$ if $\tilde{n} > 2$ (or $1 < p < \infty$ if $\tilde{n} \leq 2$), then the only nonnegative solution for (1.6) is $u \equiv 0$.*

Quaas and Sirakov in [20] consider the non-existence results for the elliptic system with Pucci extremal operators in the Euclidean space and a half space, which are essential in getting a priori bound and existence by fixed point theorem for fully nonlinear elliptic system. Motivated by the work [6], they characterized the range of powers p, q for the nonexistence of positive supersolutions of (1.7) in the Euclidean space.

Lemma 1.3 *Let $\lambda_1, \lambda_2, \Lambda_1, \Lambda_2 > 0$. Set*

$$\mathcal{M}^+_l(D^2u_l) = \mathcal{M}^+_{\lambda_l,\Lambda_l}(D^2u_l)$$

for $l = 1, 2$. Define

$$\rho_l = \frac{\lambda_l}{\Lambda_l}, \quad N_l = \rho_l(n-1) + 1.$$

Let $N_1, N_2 > 2$ and $pq > 1$ with $p, q \geq 1$. Then there are no positive supersolutions for

$$\begin{cases} \mathcal{M}^+_1(D^2u_1) + u_2^p = 0 & \text{in } \mathbb{R}^n, \\ \mathcal{M}^+_2(D^2u_2) + u_1^q = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.7}$$

if

$$\frac{2(p+1)}{pq-1} \geq N_1 - 2, \ \text{or} \ \frac{2(q+1)}{pq-1} \geq N_2 - 2.$$

By the moving plane method and Lemma 1.3 in the Euclidean space, the following Liouville-type theorem in a half space is also established under the boundedness assumption in [20].

Lemma 1.4 *Let $N_1, N_2 > 2$ and $pq > 1$ with $p, q \geq 1$. There exist no positive bounded solutions for the elliptic equation system*

$$\begin{cases} \mathcal{M}_1^+(D^2u_1) + u_2^p = 0 & \text{in } \mathbb{R}_+^n, \\ \mathcal{M}_2^+(D^2u_2) + u_1^q = 0 & \text{in } \mathbb{R}_+^n, \\ u_1 = u_2 = 0 & \text{on } \partial\mathbb{R}_+^n, \end{cases} \tag{1.8}$$

provided

$$\frac{2(p+1)}{pq-1} \geq N_1 - 2, \ \text{or} \ \frac{2(q+1)}{pq-1} \geq N_2 - 2. \tag{1.9}$$

We are also able to get rid of the boundedness assumption in the above lemma by choosing appropriate rescaling functions and employing the Doubling Lemma argument. More precisely, we prove the following

Theorem 1.2 *There exist no positive solutions for (1.8) if $p, q > 1$ and the assumption (1.9) is satisfied.*

With the Liouville-type theorem for the Euclidean space in hand and the Doubling Lemma, we can further investigate the singularity and decay estimates for positive solutions of fully nonlinear elliptic equations in a bounded domain or an exterior domain. Let $1 < p \leq p^+$ if $\tilde{n} > 2$ or $1 < p < \infty$ if $\tilde{n} \leq 2$. Recall that $\tilde{n} = \frac{\lambda}{\Lambda}(n-1) + 1$. We consider

$$\mathcal{M}_{\lambda,\Lambda}^+(D^2u) + u^p = 0 \qquad \text{in } \Omega. \tag{1.10}$$

We will establish the following

Theorem 1.3 *Let $\Omega \neq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. There exists $C = C(n,p) > 0$ such that any nonnegative solution of (1.10) satisfies*

$$u + |\nabla u|^{\frac{2}{p+1}} \leq C dist^{\frac{-2}{p-1}}(x, \partial\Omega), \qquad \forall\, x \in \Omega. \tag{1.11}$$

In particular, if Ω is an exterior domain, i.e. the set $\{x \in \mathbb{R}^n \| x| > R\}$ for some $R > 0$, then

$$u + |\nabla u|^{\frac{2}{p+1}} \leq C|x|^{\frac{-2}{p-1}}, \qquad \forall\, |x| \geq 2R.$$

If there exists a solution for a general continuous function $f(u)$, i.e. u is a nonnegative solution for

$$\mathcal{M}_{\lambda,\Lambda}^+(D^2u) + f(u) = 0 \quad \text{in } \Omega. \tag{1.12}$$

Similar singular and decay estimates also hold. Namely, if $1 < p \leq p^+$ for $\tilde{n} > 2$ or $1 < p < \infty$ for $\tilde{n} \leq 2$, we have the following corollary.

Corollary 1.1 *Assume that*

$$\lim_{u\to\infty} u^{-p} f(u) = \gamma \in (0,\infty).$$

There exists $C(n, f) > 0$ *independent of* Ω *such that any positive solution in (1.12) satisfies*

$$u + |\nabla u|^{\frac{2}{p+1}} \le C(1 + dist^{\frac{-2}{p-1}}(x, \partial\Omega)), \quad \forall\, x \in \Omega.$$

In particular, if $\Omega = \mathbb{B}_R \backslash \{0\}$ *for some R, then*

$$u + |\nabla u|^{\frac{2}{p+1}} \le C(1 + |x|^{\frac{-2}{p-1}}), \quad \forall\, 0 < |x| \le R/2.$$

Remark 1.1 Similar results also hold for $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)$ and its system in Theorem 1.1, Theorem 1.2 and Theorem 1.3.

The study of the supersolutions for

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) + u^p = 0 \quad \text{in } \mathbb{R}^n_+ \tag{1.13}$$

without assumed boundary condition is more involved. Recently, Armstrong and Sirakov [1] devised a general method for the nonexistence of positive supersolutions of elliptic operators in the whole Euclidean space and in exterior domains, which only needs the maximum principle and an asymptotically homogeneous subsolution at infinity for these elliptic operators. Notice that these elliptic operators include Pucci extremal operators as special cases. Their method also adapts to cones, in particular half spaces, for fully nonlinear operators, although the Laplacian operator is considered there. See the proof of Theorem 5.1 in [1]. Especially the optimal range of p for Liouville-type property in (1.13) could be characterized by the proof of Theorem 5.1 in [1] and the work in [3]. Leoni [14] obtains an explicit range for the Liouville-type results in (1.13), that is, there does not exist any positive solution in (1.13) for $-1 \le p \le \frac{\Lambda n+\lambda}{\Lambda n-\lambda}$. Notice that this range may not be optimal. By explicit test functions, there does exist a supersolution for $p > \frac{\Lambda(n-1)+2\lambda}{\Lambda(n-1)}$. Motivated by the work in [1], the author in [14] also points out that the inequality

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p \le 0 \qquad \text{in } \mathbb{R}^n_+ \tag{1.14}$$

does not have any positive solution for

$$-1 \le p \le \frac{\tilde{n}+1}{\tilde{n}-1}.$$

We consider the supersolutions for a system of fully nonlinear elliptic equations with Pucci's extremal operators in half spaces, i.e.

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + v^p = 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{M}^+_{\lambda,\Lambda}(D^2v) + u^q = 0 & \text{in } \mathbb{R}^n_+. \end{cases} \tag{1.15}$$

Note that the method in [1] has been applied to systems of elliptic equations in exterior domains. It is also valid for cones. We provide an elementary proof to obtain some explicit range of p, q for the nonexistence of supersolutions. We adapt the proof in [14]. The difficulty of Leoni' proof for (1.13) is to show the Liouville-type property holds for the limiting case $p = \frac{\Lambda n+\lambda}{\Lambda n-\lambda}$. In order to achieve this, some explicit subsolution is constructed under nontrivial calculations. Our main effort is also devoted to building such explicit subsolution for the operator $\mathcal{M}^+_{\lambda,\Lambda}$ instead of $\mathcal{M}^-_{\lambda,\Lambda}$. We show the following Liouville-type theorem:

Theorem 1.4 *Assume that $\tilde{n} \geq 2$ and $p, q > 0$, there exists only trivial nonnegative supersolution for (1.15) provided*

(1) $pq > 1$ and $\frac{2(p+1)}{pq-1} > \tilde{n} - 1$ or $\frac{2(q+1)}{pq-1} > \tilde{n} - 1$,

or

(2) $\frac{2(p+1)}{pq-1} = \tilde{n} - 1$ and $\frac{2(q+1)}{pq-1} = \tilde{n} - 1$,

or

(3) $pq = 1$.

Combining our idea in Theorem 1.4 and the estimates for $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)$ in [14], we are able to establish the following Liouville-type results for

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2u) + v^p = 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2v) + u^q = 0 & \text{in } \mathbb{R}^n_+. \end{cases} \tag{1.16}$$

Corollary 1.2 *There exists only trivial nonnegative supersolution for (1.16) if*

(1) $pq > 1$ and $\frac{2(p+1)}{pq-1} > \frac{\Lambda n}{\lambda} - 1$ or $\frac{2(q+1)}{pq-1} > \frac{\Lambda n}{\lambda} - 1$,

or

(2)$\frac{2(p+1)}{pq-1} = \frac{\Lambda n}{\lambda} - 1$ and $\frac{2(q+1)}{pq-1} = \frac{\Lambda n}{\lambda} - 1$,

or

(3) $pq = 1$.

Finally we note that there is a large literature concerning Liouville-type results for solution (or supersolution) of elliptic equations or system. We make no attempt to create an exhaustive bibliography here. We refer to [2], [5], [7], [9], [11], [13], [16], [21] and references therein for more account.

The outline of the paper is as follows. In Section 2, we present the basic results for the definition of viscosity solution, comparison principle, Doubling Lemma and so on. Section 3 is devoted to the proof of removing the boundedness assumption for fully nonlinear elliptic equations and systems. We also show the singularity and decay estimates for a single equation. The Liouville-type theorem for a system of equations in a half space without boundary assumption is considered in Section 4. Throughout the paper, C and C_1 denote generic positive constants, which are independent of u, v and may vary from line to line.

2 Preliminaries

In this section we collect some basic results which will be applied throughout the paper for fully nonlinear elliptic equations. We refer to [4], [6], [19] and references therein for the proofs and results.

Let us recall the notion of viscosity sub and supersolutions of fully nonlinear elliptic equations

$$F(x,u,D^2u)=0 \quad \text{in } \Omega, \tag{2.1}$$

where Ω is an open domain in $\mathbb{R}^n$ and $F:\Omega\times\mathbb{R}\times S_n\to\mathbb{R}$ is a continuous map with $F(x,t,M)$ satisfying (1.3) for every fixed $t\in\mathbb{R}$, $x\in\Omega$.

Definition: A continuous function $u:\Omega\to\mathbb{R}$ is a viscosity supersolution (subsolution) of (2.1) in Ω, when the following condition holds: If $x_0\in\Omega$, $\phi\in C^2(\Omega)$ and $u-\phi$ has a local minimum (maximum) at x_0, then

$$F(x_0,u(x_0),D^2\phi(x_0))\le(\ge)0.$$

If u is a viscosity supersolution (subsolution), we say that u verifies

$$F(x,u,D^2u)\le(\ge)0$$

in the viscosity sense.

We say that u is a viscosity solution of (2.1) when it simultaneously is a viscosity subsolution and supersolution.

We will make use of the following comparison principle (see e.g. [6]).

Lemma 2.1 *(Comparison Principle) Let $\Omega\in\mathbb{R}^n$ be a bounded domain and $f\in C(\Omega)$. If u and v are respectively a supersolution and subsolution either of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u)=f(x)$ or of $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)=f(x)$ in Ω, and $u\ge v$ on $\partial\Omega$, then $u\ge v$ in $\bar\Omega$.*

The following version of the Hopf boundary lemma holds (see e.g. [19]).

Lemma 2.2 *Let Ω be a regular domain and $u\in W^{2,n}_{loc}(\Omega)\cap C(\bar\Omega)$ be a nonnegative solution to*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u)+c(x)u\le 0 \quad \textit{in } \Omega$$

with bounded $c(x)$. Then either $u\equiv 0$ in Ω or $u(x)>0$ for all $x\in\Omega$. Moreover, in the latter case for any $x\in\partial\Omega$ such that $u(x_0)=0$,

$$\limsup_{t\to0^+}\frac{u(x_0-t\nu)-u(x_0)}{t}<0,$$

where ν is the outer normal to $\partial\Omega$.

We are going to use the following regularity results in [4] for Pucci operators in the blow-up argument.

Lemma 2.3 *(Regularity Lemma) If u is a viscosity solution to the fully nonlinear elliptic equation with Pucci extremal operator*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u)+g(x)=0 \tag{2.2}$$

in a ball $\mathbb{B}_{2R}$ and $g\in L^p(\mathbb{B}_R)$ for some $p\ge n$, then $u\in W^{2,p}(\mathbb{B}_R)$ and the following interior estimate holds

$$\|u\|_{W^{2,p}(\mathbb{B}_R)}\le C(\|u\|_{L^\infty(\mathbb{B}_{2R})}+\|g\|_{L^p(\mathbb{B}_{2R})}). \tag{2.3}$$

Furthermore, if $g \in C^\alpha$ for some $\alpha \in (0, 1)$, then $u \in C^{2,\alpha}$ and

$$\|u\|_{C^{2,\alpha}(\mathbb{B}_R)} \le C(\|u\|_{L^\infty(\mathbb{B}_{2R})} + \|g\|_{C^\alpha(\mathbb{B}_{2R})}). \tag{2.4}$$

In addition, if (2.2) holds in a regular domain and $u = 0$ on the boundary, then u satisfies a C^α- estimate up to the boundary.

Note that the above $C^{2,\alpha}$ estimate depends on the convexity of the Pucci extremal operator. Next we state the closeness of a family of viscosity solutions to fully nonlinear equations (see e.g. [4]).

Lemma 2.4 *Assume u_n and g_n are sequences of continuous functions and u_n is a solution (or subsolution, or supersolution) of the equation*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_n) + g_n(x) = 0 \quad \text{in } \Omega.$$

Assume that u_n and g_n converge uniformly on compact subsets of Ω to function u and g. Then u is a solution (or subsolution, or supersolution) of the equation

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + g(x) = 0 \quad \text{in } \Omega.$$

We state the following technical lemma that is frequently used in Section 3. The proof of this lemma is given in [18]. An interested reader may refer to it for more details. Based on the doubling property, we can start the rescaling process to prove local estimates of solutions for fully nonlinear equations.

Lemma 2.5 *(Doubling lemma) Let (X, d) be a complete metric space and $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Define $M : D \to (0, \infty)$ to be bounded on compact subsets of D. If $y \in D$ is such that*

$$M(y)dist(y, \Gamma) > 2k$$

for a fixed positive number k, where $\Gamma = \Sigma \setminus D$, then there exists $x \in D$ such that

$$M(x)dist(x, \Gamma) > 2k, \quad M(x) \ge M(y).$$

Moreover,

$$M(z) \le 2M(x), \quad \forall z \in D \cap \bar{B}(x, kM^{-1}(x)).$$

Remark 2.1 If $\Gamma = \emptyset$, then $dist(x, \Gamma) := \infty$. In this case, we have following the version of the Doubling Lemma. Let $D = \Sigma \subset X$, with Σ closed. Define $M : D \to (0, \infty)$ to be bounded on compact subsets of D, For every $y \in D$, there exists $x \in D$ such that

$$M(x) \ge M(y)$$

and

$$M(z) \le 2M(x), \quad \forall z \in D \cap \bar{B}(x, kM^{-1}(x)).$$

3 Liouville-type theorems for elliptic equations in half spaces

We first present the proof of Theorem 1.1. Our idea is the combination of doubling property and blow-up argument. This idea seems to be powerful in getting rid of the boundedness assumption whenever proving Liouville-type theorems. We refer to [15] for applications of this idea in higher order elliptic equations.

Proof of Theorem 1.1. Suppose that a solution u to the equation (1.6) is unbounded. Namely, there exists a sequence of $(y_k) \in \mathbb{R}^n_+$ such that

$$u(y_k) \to \infty$$

as $k \to \infty$. Set

$$M(y) := u^{\frac{p-1}{2}}(y) : \mathbb{R}^n_+ \to \mathbb{R}.$$

Then $M(y_k) \to \infty$ as $k \to \infty$ by the fact that $p > 1$. By taking $D = \Sigma = X = \overline{\mathbb{R}^n_+}$ in the Doubling Lemma (i.e. Lemma 2.5) and Remark 2.1, there exists another sequence of $\{x_k\}$ such that

$$M(x_k) \geq M(y_k)$$

and

$$M(z) \leq 2M(x_k), \quad \forall z \in B_{k/M(x_k)}(x_k) \cap \overline{\mathbb{R}^n_+}.$$

Set

$$d_k := x_{k,n} M(x_k),$$

where $x_k = (x_{k,1}, \cdots, x_{k,n})$ and

$$H_k := \{\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n | \xi_n > -d_k\}.$$

We define a new function

$$v_k(\xi) := \frac{u(x_k + \frac{\xi}{M(x_k)})}{M^{\frac{2}{p-1}}(x_k)}.$$

Then, $v_k(\xi)$ is the nonnegative solution of

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 v_k) + v_k^p = 0 & \text{in } H_k, \\ v_k = 0 & \text{on } \partial H_k = \{\xi \in \mathbb{R}^n | \xi_n = -d_k\} \end{cases} \tag{3.1}$$

with

$$v_k^{\frac{p-1}{2}}(0) = 1 \tag{3.2}$$

and

$$v_k^{\frac{p-1}{2}}(\xi) \leq 2, \qquad \forall \xi \in H_k \cap B_k(0). \tag{3.3}$$

Two cases may occur as $k \to \infty$, either Case (1)

$$x_{k,n} M(x_k) \to \infty$$

for a subsequence still denoted as before, or Case (2)

$$x_{k,n} M(x_k) \to d$$

for a subsequence still denoted as before, here $d \geq 0$. If Case (1) occurs, i.e. $H_k \cap B_k(0) \to \mathbb{R}^n$ as $k \to \infty$, then for any smooth compact set D in $\mathbb{R}^n$, there exists k_0 large enough such that $D \subset (H_k \cap B_k(0))$ as $k \geq k_0$. By regularity lemma (i.e. Lemma 2.3), (3.3) and Arzelá-Ascoli theorem, $v_k \to v$ in $C^2(\bar{D})$ for a subsequence. Furthermore, using a diagonalization argument, $v_k \to v$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$. From Lemma 2.4, we know that v solves

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + v^p = 0 \quad \text{in } \mathbb{R}^n.$$

Thanks to Lemma 1.1, there exists only a trivial solution provided

$$1 < p \leq p^+ \quad \text{for } \lambda(n-1) > \Lambda \tag{3.4}$$

or

$$1 < p < \infty \quad \text{for } \lambda(n-1) \leq \Lambda. \tag{3.5}$$

In the above, we have used the fact that $\tilde{n} = 2$ is equivalent to $\lambda(n-1) = \Lambda$. However, (3.2) implies that

$$v^{\frac{p-1}{2}}(0) = 1,$$

which indicates that v is nontrivial. This contradiction leads to the conclusion that u in (1.6) is bounded in the above range of p.

If the Case (2) occurs, we make a further translation. Set

$$\tilde{v}_k(\xi) := v_k(\xi - d_k e_n) \quad \text{for } \xi \in \overline{\mathbb{R}^n_+}.$$

Then $\tilde{v}_k$ satisfies

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 \tilde{v}_k) + \tilde{v}_k^p = 0 & \text{in } \mathbb{R}^n_+, \\ \tilde{v}_k \geq 0 & \text{in } \mathbb{R}^n_+, \\ \tilde{v}_k = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.6}$$

While

$$\tilde{v}_k^{\frac{p-1}{2}}(d_k e_n) = 1 \tag{3.7}$$

and

$$\tilde{v}_k^{\frac{p-1}{2}}(\xi) \leq 2, \qquad \forall \xi \in \mathbb{R}^n_+ \cap B_k(d_k e_n). \tag{3.8}$$

For any smooth compact D in $\overline{\mathbb{R}^n_+}$, there also exists k_0 large enough such that $D \subset (\overline{\mathbb{R}^n_+} \cap B_k(0))$ for any $k \geq k_0$. Thanks to regularity Lemma 2.3 and (3.8), we can extract a subsequence of $\tilde{v}_k$ such that $\tilde{v}_k \to v$ in $C^2(\bar{D}) \cap C(\bar{D})$. A diagonalization argument shows that $\tilde{v}_k \to v$ uniformly as $k \to \infty$. Furthermore, by Lemma 2.4, v solves

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + v^p = 0 & \text{in } \mathbb{R}^n_+, \\ v \geq 0 & \text{in } \mathbb{R}^n_+, \\ v = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.9}$$

Due to Lemma 1.2, we readily have that $v \equiv 0$ if

$$1 < p \leq \tilde{p}^+ \quad \text{for } \lambda(n-2) > \Lambda \tag{3.10}$$

or

$$1 < p < \infty \quad \text{for } \lambda(n-2) \le \Lambda. \tag{3.11}$$

It contradicts again with the fact that

$$v(de_n)^{\frac{p-1}{2}} = 1 \tag{3.12}$$

from (3.7). Hence u is bounded in Case (2).

Together with (3.4), (3.5), (3.10) and (3.11), we infer that u is bounded in (1.6) if $1 < p \le p^+$ in the case of $\lambda(n-1) > \Lambda$ or if $1 < p < \infty$ in the case of $\lambda(n-1) \le \Lambda$. Note again that $\tilde{n} = 2$ implies that $\lambda(n-1) = \Lambda$. Applying Lemma 1.2 again, we obtain Theorem 1.1 in the above range of p.

We are now in the position to prove Theorem 1.2. Since we consider the elliptic system with different powers p, q, we shall choose the rescaling function appropriately.

Proof of Theorem 1.2. Assume by contradiction that either u_1 or u_2 is unbounded, that is, there exists a sequence y_k such that

$$M_k(y_k) = u_1^{1/\alpha}(y_k) + u_2^{1/\beta}(y_k) \to \infty$$

as $k \to \infty$. The constant α,β are positive numbers which will be determined later. From the Doubling Lemma and Remark 1, there exists a sequence of $\{x_k\}$ such that

$$M(x_k) \ge M(y_k)$$

and

$$M(z) \le 2M(x_k), \quad \forall z \in B_{k/M(x_k)}(x_k) \cap \overline{\mathbb{R}^n_+}.$$

Define

$$d_k := x_{k,n} M(x_k)$$

and

$$H_k := \{\xi \in \mathbb{R}^n | \xi_n > -d_k\}.$$

We do the following rescaling,

$$v_{1,k}(\xi) := \frac{u_1(x_k + \frac{\xi}{M(x_k)})}{M^\alpha(x_k)},$$

$$v_{2,k}(\xi) := \frac{u_2(x_k + \frac{\xi}{M(x_k)})}{M^\beta(x_k)}.$$

Then, by (1.8), $v_{1,k}(\xi)$, $v_{2,k}(\xi)$ satisfy

$$\begin{cases} \mathcal{M}_1^+(D^2 v_{1,k}) M_k^{\alpha+2}(x_k) + M_k^{p\beta}(x_k) v_{2,k}^p = 0 & \text{in } H_k, \\ \mathcal{M}_2^+(D^2 v_{2,k}) M_k^{\beta+2}(x_k) + M_k^{q\alpha}(x_k) v_{1,k}^q = 0 & \text{in } H_k, \\ v_{1,k} = v_{2,k} = 0 & \text{in } \partial H_k. \end{cases} \tag{3.13}$$

In order to get rid of $M_k(x_k)$ in (3.13), by setting $\alpha + 2 = p\beta$ and $\beta + 2 = q\alpha$, we conclude that

$$\alpha = \frac{2(p+1)}{pq-1},$$
$$\beta = \frac{2(q+1)}{pq-1}.$$

With so chosen α, β, then $v_{1,k}, v_{2,k}$ solve

$$\begin{cases} \mathcal{M}_1^+(D^2 v_{1,k}) + v_{2,k}^p = 0 & \text{in } H_k, \\ \mathcal{M}_2^+(D^2 v_{2,k}) + v_{1,k}^q = 0 & \text{in } H_k. \\ v_{1,k} = v_{2,k} = 0 & \text{in } \partial H_k. \end{cases} \tag{3.14}$$

Furthermore,

$$v_{1,k}^{\frac{1}{\alpha}}(0) + v_{2,k}^{\frac{1}{\beta}}(0) = 1 \tag{3.15}$$

and

$$v_{1,k}^{\frac{1}{\alpha}}(\xi) + v_{2,k}^{\frac{1}{\beta}}(\xi) \leq 2, \qquad \forall \xi \in H_k \cap \mathbb{B}_k(0).$$

Two cases may occur as $k \to \infty$, either Case (1),

$$d_k \to \infty$$

for a subsequence still denoted as before, or Case (2)

$$d_k \to d$$

for a subsequence still denoted as before. We note that $d \geq 0$.

If Case (1) occurs, i.e. $H_k \cap \mathbb{B}_k(0) \to \mathbb{R}^n$, we argue similarly as in the proof of Theorem 1.1. For any smooth compact set D in $\mathbb{R}^n$, by Lemma 2.3 and Arzelá-Ascoli theorem, we know that $v_{1,k} \to v_1$ and $v_{2,k} \to v_2$ in $C^2(\bar{D})$ for a subsequence. Using a diagonalization argument, $v_{1,k} \to v_1$ and $v_{2,k} \to v_2$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$. From Lemma 2.4, we obtain that v_1, v_2 satisfy

$$\begin{cases} \mathcal{M}_1^+(D^2 v_1) + v_2^p = 0 & \text{in } \mathbb{R}^n, \\ \mathcal{M}_2^+(D^2 v_2) + v_1^q = 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{3.16}$$

As shown in Lemma 1.3, $v_1 \equiv v_2 \equiv 0$ provided

$$\frac{2(p+1)}{pq-1} \geq N_1 - 2, \text{ or } \frac{2(q+1)}{pq-1} \geq N_2 - 2.$$

Nevertheless, (3.15) indicates that either v_1 or v_2 is nontrivial. We arrive at the contradiction, which indicates u_1, u_2 in (1.8) are actually bounded in Case (1).

If Case (2) occurs, we translate the equation to be in the standard half space. Let

$$\tilde{v}_{1,k}(\xi) := v_{1,k}(\xi - d_k e_n) \quad \text{for } \xi \in \overline{\mathbb{R}^n_+},$$

$$\tilde{v}_{2,k}(\xi) := v_{2,k}(\xi - d_k e_n) \quad \text{for } \xi \in \overline{\mathbb{R}^n_+}.$$

Then $\tilde v_{1,k}$, $\tilde v_{2,k}$ satisfy

$$\begin{cases} \mathcal{M}_1^+(D^2\tilde v_{1,k}) + \tilde v_{2,k}^p = 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{M}_2^+(D^2\tilde v_{2,k}) + \tilde v_{1,k}^q = 0 & \text{in } \mathbb{R}^n_+, \\ \tilde v_{1,k} = \tilde v_{1,k} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.17}$$

Moreover,

$$\tilde v_{1,k}^{\frac{1}{\alpha}}(d_k e_n) + \tilde v_{2,k}^{\frac{1}{\beta}}(d_k e_n) = 1 \tag{3.18}$$

and

$$\tilde v_{1,k}^{\frac{1}{\alpha}}(\xi) + \tilde v_{2,k}^{\frac{1}{\beta}}(\xi) \le 2, \qquad \forall \xi \in \mathbb{R}^n_+ \cap B_k(d_k e_n). \tag{3.19}$$

Similar argument as in the proof of Theorem 1.1 shows that there exist $\tilde v_{1,k}$ and $\tilde v_{2,k}$ such that

$$\tilde v_{1,k} \to \tilde v_1$$

and

$$\tilde v_{2,k} \to \tilde v_2$$

in $C^2_{loc}(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ as $k \to \infty$. $\tilde v_1$ and $\tilde v_2$ solve

$$\begin{cases} \mathcal{M}_1^+(D^2\tilde v_1) + \tilde v_2^p = 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{M}_2^+(D^2\tilde v_2) + \tilde v_1^q = 0 & \text{in } \mathbb{R}^n_+, \\ \tilde v_1 = \tilde v_1 = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.20}$$

Lemma 1.4 and (3.19) yield that $\tilde v_1 \equiv \tilde v_2 \equiv 0$ when (1.9) holds. However, it contradicts to the fact of (3.18).

In conclusion, we obtain that u is bounded in (1.8) when the exponents p and q satisfy (1.9). From Lemma 1.4 again, we conclude that the boundedness assumption is not essential, i.e. Theorem 1.2 holds.

With the help of Lemma 1.1 and the Doubling Lemma, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. We again argue by contradiction. Suppose that (1.11) is false. Then, there exists a sequence of functions u_k in (1.10) on Ω_k such that

$$M_k = u_k^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}}$$

satisfying

$$M_k(y_k) > 2k dist^{-1}(y_k, \partial\Omega_k).$$

By the Doubling Lemma, there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) \ge M_k(y_k),$$

$$M_k(x_k) > 2k dist^{-1}(x_k, \partial\Omega_k)$$

and

$$M_k(z) \le 2M_k(x_k), \quad \text{if } |z - x_k| \le k M_k^{-1}(x_k).$$

We introduce a rescaled function

$$v_k(\xi) = \frac{u_k(x_k + \frac{\xi}{M_k(x_k)})}{M_k^{\frac{2}{p-1}}}.$$

Simple calculation yields that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v_k) + v_k^p = 0, \qquad \forall |\xi| \leq k. \tag{3.21}$$

Moreover,

$$(v_k^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}})(0) = 1 \tag{3.22}$$

and

$$(v_k^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}})(\xi) \leq 2, \qquad \forall |\xi| \leq k. \tag{3.23}$$

For any smooth compact set D in $\mathbb{R}^n$, there exists k_0 large enough such that $D \subset \mathbb{B}_k(0)$ as $k \geq k_0$. By Lemma 2.3 and (3.23), we have

$$\|v_k\|_{C^{2,\alpha}(D)} \leq C$$

for some $C > 0$. From Arzelá-Ascoli theorem, up to a subsequence, $v_k \to v$ in $C^2(\bar{D})$. In addition, by a diagonalization argument and Lemma 2.4, $v_k \to v$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$, which solves

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + v^p = 0 \quad \text{in } \mathbb{R}^n.$$

Since $1 < p \leq p^+$, Lemma 1.1 implies that the only solution is $v \equiv 0$. However, (3.22) shows that v is impossible to be trivial. Therefore, this contradiction leads to the conclusion in Theorem 1.3.

For the proof of Corollary 1.1, it is very similar to the above argument. We shall omit it here. The interested reader may refer to the above proof and [18].

4 A Liouville-type theorem for supersolutions of elliptic systems in a half space

We introduce the following algebraic result in [14] for the eigenvalue of a special symmetric matrix.

Lemma 4.1 *Let $\nu, \omega \in \mathbb{R}^n$ be unitary vectors and a_1, a_2, a_3 and a_4 be constants. For the symmetric matrix,*

$$A = a_1 \nu \otimes \nu + a_2 \omega \otimes \omega + a_3(\nu \otimes \omega + \omega \otimes \nu) + a_4 I_n,$$

where $\nu \otimes \omega$ denotes the $n \times n$ matrix whose i, j entry is $\nu_i \omega_j$, the eigenvalues of A are given as follows,

•a_4, *with multiplicity (at least) $n - 2$.*

$\bullet a_4 + \frac{a_1+a_2+2a_3\nu\cdot\omega\pm\sqrt{(a_1+a_2+2a_3\nu\cdot\omega)^2+4(1-(\nu\cdot\omega))^2(a_3^2-a_1a_2)^2}}{2}$, *which are simple (if different from a_4).*

In particular, if either $a_3^2 = a_1a_2$ or $(\nu\cdot\omega)^2 = 1$, then the eigenvalues are a_4 with multiplicity $n-1$ and $a_4 + a_1 + a_2 + 2a_3\nu\cdot\omega$, which is simple.

Let us consider a lower semicontinuous function $u \in \overline{\mathbb{R}^n_+} \to [0, \infty)$ for

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \le 0 \quad \text{in } \mathbb{R}^n_+ \tag{4.1}$$

in viscosity sense. For any $r > 0$, we define the function

$$m_u(r) = \inf_{\mathbb{B}^+_r} \frac{u(x)}{x_n}, \tag{4.2}$$

where $\mathbb{B}^+_r$ is the half ball centered at the origin with radius r in $\mathbb{R}^n_+$. We present the following three – circles Hadamard type results for superharmonic functions in [14].

Lemma 4.2 *Let $u \in \overline{\mathbb{R}^n_+} \to [0, \infty)$ be a lower semicontinuous function satisfying (4.1). Then the function $m_u(r)$ in (4.2) is a concave function of $r^{-\tilde{n}}$, i.e. for every fixed $R > r > 0$ and for all $r \le \rho \le R$, one has*

$$m_u(\rho) \ge \frac{m_u(r)(\rho^{-\tilde{n}} - R^{-\tilde{n}}) + m_u(R)(r^{-\tilde{n}} - \rho^{-\tilde{n}})}{r^{-\tilde{n}} - R^{-\tilde{n}}}. \tag{4.3}$$

Consequently,

$$r \in (0, \infty) \to m_u(r)r^{\tilde{n}}$$

is nondecreasing.

To prove the Liouville-type theorem in (1.15) for the critical case

$$\frac{2(p+1)}{pq-1} = \tilde{n} - 1, \text{ and } \frac{2(q+1)}{pq-1} = \tilde{n} - 1,$$

we will compare the supersolutions u, v with an explicit subsolution of the equation

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) = (\frac{x_n}{|x|^{\tilde{n}}})^{\frac{\tilde{n}+1}{\tilde{n}-1}}.$$

Such a subsolution is constructed as follows.

Lemma 4.3 *There exist positive constants $e, f > 0$ and $r_0 \ge 1$, which only depend on λ, Λ and n such that the function*

$$\Gamma(x) = \frac{x_n}{|x|^{\tilde{n}}}(eln|x| + f(\frac{x_n}{|x|})^2)$$

satisfies

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma) \le (\frac{x_n}{|x|^{\tilde{n}}})^{\frac{\tilde{n}+1}{\tilde{n}-1}} \quad \textit{in } \mathbb{R}^n_+\backslash\mathbb{B}_{r_0} \tag{4.4}$$

in the classical sense.

Proof. We consider

$$\Gamma_1(x) := \frac{x_n}{|x|^{\tilde{n}}} ln|x|$$

and

$$\Gamma_2(x) := \frac{x_n^3}{|x|^{\tilde{n}+2}}.$$

Then $\Gamma(x) = e\Gamma_1(x) + f\Gamma_2(x)$. From the property of the Pucci maximal operator, it yields that

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma) \le -e\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma_1) - f\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma_2). \tag{4.5}$$

In order to obtain (4.4), we estimate the terms on the right hand side of (4.5), respectively. As far as Γ_1 is concerned, direct calculations show that

$$\begin{aligned} D^2\Gamma_1(x) = \ & \tfrac{x_n}{|x|^{\tilde{n}+2}}\{[(\tilde{n}+2)\tilde{n}ln|x| - 2(\tilde{n}+1)]\tfrac{x}{|x|}\otimes\tfrac{x}{|x|} + (1-\tilde{n}ln|x|)e_n\otimes e_n \\ & +(1-\tilde{n}ln|x|)\tfrac{|x|}{x}(\tfrac{x}{|x|}\otimes e_n + e_n\otimes\tfrac{x}{|x|}) - (\tilde{n}ln|x| - 1)I_n\}. \end{aligned}$$

Recall that $\tilde{n} = \frac{\lambda}{\Lambda}(n-1)+1$. According to Lemma 4.1, the eigenvalue $\mu_1, \mu_2, \cdots, \mu_n$ of $D^2\Gamma_1$ are

$$\mu_1 = \frac{x_n}{|x|^{\tilde{n}+2}}\frac{\tilde{n}^2ln|x| - 3\tilde{n}ln|x| - 2\tilde{n} + 3 + \sqrt{D}}{2},$$

$$\mu_2 = \frac{x_n}{|x|^{\tilde{n}+2}}\frac{\tilde{n}^2ln|x| - 3\tilde{n}ln|x| - 2\tilde{n} + 3 - \sqrt{D}}{2},$$

$$\mu_i = -\frac{x_n}{|x|^{\tilde{n}+2}}(\tilde{n}ln|x| - 1), \quad 3 \le i \le n,$$

where

$$\begin{aligned} D \ = \ & [\tilde{n}(\tilde{n}+2)ln|x| - 2(\tilde{n}+1) + 3(1-\tilde{n}ln|x|)]^2 \\ & +4(1-\tfrac{x_n^2}{|x|^2})\{(1-\tilde{n}ln|x|)^2\tfrac{|x|^2}{x_n^2} - [(\tilde{n}+2)\tilde{n}ln|x| - 2(\tilde{n}+1)](1-\tilde{n}ln|x|)\} \\ \ge \ & [(\tilde{n}+2)(\tilde{n}ln|x| - 2) + 3(1-\tilde{n}ln|x|)]^2 \\ & +4(1-\tfrac{x_n^2}{|x|^2})\{(1-\tilde{n}ln|x|)^2\tfrac{|x|^2}{x_n^2} - (\tilde{n}+2)(\tilde{n}ln|x| - 2)(1-\tilde{n}ln|x|)\} \\ \ge \ & [(\tilde{n}ln|x| - 2)(\tilde{n}-1)]^2 + 4(1-\tfrac{x_n^2}{|x|^2})(\tilde{n}ln|x| - 2)^2[\tfrac{|x|^2}{x_n^2} + (\tilde{n}+2)]. \end{aligned}$$

Hence

$$\sqrt{D} \ge (\tilde{n}ln|x| - 2)(\tilde{n}-1).$$

For $r > r_0$, it follows that $\mu_1 \geq 0$ and $\mu_i \leq 0$ for $2 \leq i \leq n$, where r_0 depends on Λ, λ and n. Therefore, one has

$$\begin{aligned}
\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma_1) &= \Lambda\mu_1 + \lambda \textstyle\sum_{i=2}^n \mu_i \\
&= \frac{x_n}{|x|^{\tilde{n}+2}}\{\frac{(\Lambda+\lambda)(\tilde{n}^2 ln|x| - 3\tilde{n} ln|x| - 2\tilde{n}+3)+(\Lambda-\lambda)\sqrt{D}}{2} \\
&\quad -(n-2)\lambda(\tilde{n} ln|x| - 1)\} \\
&\geq \frac{x_n}{|x|^{\tilde{n}+2}} \frac{(\Lambda+\lambda)(-2\tilde{n}+3)-2(\Lambda-\lambda)(\tilde{n}-1)+2(n-2)\lambda}{2} \\
&= -\frac{x_n}{|x|^{\tilde{n}+2}} \frac{2\lambda n - \Lambda - \lambda}{2} \\
&= -c_1 \frac{x_n}{|x|^{\tilde{n}+2}},
\end{aligned}$$

where $c_1 = \frac{2\lambda n - \Lambda - \lambda}{2}$. Since $\tilde{n} = \frac{\lambda}{\Lambda}(n-1) + 1 \geq 2$, we get $c_1 > 0$. By the argument in Theorem 2.3 in [14], we have

$$\begin{aligned}
\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma_2) &\geq \frac{\lambda x_n^3}{|x|^{\tilde{n}+4}}\{(\tilde{n}+2)[\tilde{n} - 3 - \frac{\Lambda}{\lambda}(n-1)] + 3(3 - \frac{\Lambda}{\lambda})\frac{|x|^2}{x_n^2}\} \\
&\geq \frac{\lambda x_n^3}{|x|^{\tilde{n}+4}}\{\tilde{n}[\tilde{n} - 3 - \frac{\Lambda}{\lambda}(n-1)] + 2[\tilde{n} - \frac{\Lambda}{\lambda}(n-1)] + 3(1 - \frac{\Lambda}{\lambda})\frac{|x|^2}{x_n^2}\} \\
&= \frac{\lambda x_n^3}{|x|^{\tilde{n}+4}}\{\tilde{n}(\frac{\lambda}{\Lambda} - \frac{\Lambda}{\lambda})(n-1) - 2\frac{\Lambda}{\lambda}(n-1) + 3(1 - \frac{\Lambda}{\lambda})\frac{|x|^2}{x_n^2}\} \\
&\geq -\frac{x_n^3}{|x|^{\tilde{n}+4}}\{c_2 - c_3\frac{|x|^2}{x_n^2}\},
\end{aligned}$$

where $c_2 = \tilde{n}(\frac{\Lambda^2-\lambda^2}{\Lambda})(n-1) + 2\Lambda(n-1)$ and $c_3 = 3(\Lambda - \lambda)$. Then setting $f = c_2^{-1}$ and $e = \frac{c_3}{c_2 c_1}$, we obtain

$$\begin{aligned}
-\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma) &\leq -e\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma_1) - f\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma_2) \\
&\leq ec_1\frac{x_n}{|x|^{\tilde{n}+2}} + fc_2\frac{x_n^3}{|x|^{\tilde{n}+4}} - fc_3\frac{x_n}{|x|^{\tilde{n}+2}} \\
&\leq \frac{x_n^3}{|x|^{\tilde{n}+4}},
\end{aligned}$$

Furthermore, since $\tilde{n} \geq 2$, a direct calculation yields that

$$\begin{aligned}
-\mathcal{M}^+_{\lambda,\Lambda}(D^2\Gamma) &\leq \frac{x_n^3}{|x|^{\tilde{n}+4}} \\
&\leq (\frac{x_n}{|x|^{\tilde{n}}})^{\frac{\tilde{n}+1}{\tilde{n}-1}}.
\end{aligned}$$

Hence the lemma is completed.

Proof of Theorem 1.4. By the strong maximal principle (i.e. Lemma 2.2), we may assume that $u, v > 0$ in $\mathbb{R}^n_+$. Let us rescale the supersolutions in (1.15). For every $r > 0$, we set

$$u_r(x) = u(rx),$$

$$v_r(x) = v(rx).$$

Then $u_r, v_r > 0$ are supersolutions for

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u_r) + r^2v_r^p = 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{M}^+_{\lambda,\Lambda}(D^2v_r) + r^2u_r^q = 0 & \text{in } \mathbb{R}^n_+. \end{cases} \tag{4.6}$$

Next we will choose appropriate test functions for supersolutions u_r, v_r. Selecting a smooth, concave, nonincreasing function: $\eta : [0,\ +\infty) \to R$ satisfying

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1/2, \\ > 0 & \text{for } \ 1/2 < t < 3/4, \\ \le 0 & \text{for } t \ge 3/4. \end{cases} \tag{4.7}$$

Fix a point $a = (0, 1)$. Here $\mathbb{B}_r(a)$ is a ball centered at a with radius r. Let

$$U(x) = (\inf_{\mathbb{B}_{1/2}(a)} u_r)\eta(|x - a|),$$

$$V(x) = (\inf_{\mathbb{B}_{1/2}(a)} v_r)\eta(|x - a|).$$

It is easy to see that $u_r \ge U$ in $\overline{\mathbb{B}_{1/2}}(a)$, $u_r = U$ at some point on $\partial\mathbb{B}_{1/2}(a)$ by the maximum principle (i.e. Lemma 2.1) and $u_r > U$ outside $\mathbb{B}_{3/4}(a)$. By the same observation, $v_r \ge V$ in $\overline{\mathbb{B}_{1/2}}(a)$, $v_r = V$ at some point on $\partial\mathbb{B}_{1/2}(a)$ and $v_r > V$ outside $\mathbb{B}_{3/4}(a)$. Therefore, the infimum of $u_r - U, v_r - V$ is non-positive and achieved at x_1, x_2 in $\mathbb{B}_{3/4}(a)\backslash\mathbb{B}_{1/2}(a)$, respectively. From the definition of a viscosity solution and taking into account that U, V are test functions for u_r, v_r, respectively, it yields that

$$v_r^p(x_1) \le \frac{C_1}{r^2} \inf_{\mathbb{B}_{1/2}(a)} u_r \tag{4.8}$$

and

$$u_r^q(x_2) \le \frac{C_1}{r^2} \inf_{\mathbb{B}_{1/2}(a)} v_r, \tag{4.9}$$

where

$$C_1 = \sup_{\mathbb{B}_{3/4}(a)} (-\mathcal{M}^+_{\lambda,\Lambda}(D^2\eta)) = \sup_{\mathbb{B}_{3/4}(a)} (-\lambda\triangle\eta) = -\lambda \inf_{t\in[1/2,\ 3/4]} (\eta''(t) + (n-1)t^{-1}\eta').$$

Since $u_r(x)$ and $v_r(x)$ are also supersolutions for $\mathcal{M}^+_{\lambda,\Lambda}(D^2u_r) = 0$ and $\mathcal{M}^+_{\lambda,\Lambda}(D^2v_r) = 0$, respectively, the monotonicity property (see [6]) implies that

$$\inf_{\mathbb{B}_{1/2}(a)} u_r \le C \inf_{\mathbb{B}_{3/4}(a)} u_r, \tag{4.10}$$

$$\inf_{\mathbb{B}_{1/2}(a)} v_r \leq C \inf_{\mathbb{B}_{3/4}(a)} v_r. \tag{4.11}$$

Furthermore, From (4.8)-(4.11), we get

$$\begin{aligned}
(\inf_{\mathbb{B}_{3/4}(a)} v_r)^p &\leq v_r^p(x_1) \leq \frac{C_1}{r^2} \inf_{\mathbb{B}_{1/2}(a)} u_r \leq \frac{C}{r^2} \inf_{\mathbb{B}_{3/4}(a)} u_r \leq \frac{C}{r^2}(\frac{C_1}{r^2} \inf_{\mathbb{B}_{1/2}(a)} v_r)^{\frac{1}{q}} \\
&\leq \frac{C}{r^{2(1+\frac{1}{q})}}(\inf_{\mathbb{B}_{3/4}(a)} v_r)^{\frac{1}{q}},
\end{aligned}$$

that is,

$$(\inf_{\mathbb{B}_{3/4}(a)} v_r) \leq \frac{C}{r^{\frac{2(q+1)}{pq-1}}}. \tag{4.12}$$

Similar argument indicates that

$$\begin{aligned}
(\inf_{\mathbb{B}_{3/4}(a)} u_r)^q \leq u_r^q(x_1) &\leq \frac{C_1}{r^2} \inf_{\mathbb{B}_{1/2}(a)} v_r \leq \frac{C}{r^2} \inf_{\mathbb{B}_{3/4}(a)} v_r \leq \frac{C}{r^2}(\frac{C_1}{r^2} \inf_{\mathbb{B}_{1/2}(a)} u_r)^{\frac{1}{p}} \\
&\leq \frac{C}{r^{2(1+\frac{1}{p})}}(\inf_{\mathbb{B}_{3/4}(a)} u_r)^{\frac{1}{p}},
\end{aligned}$$

that is,

$$(\inf_{\mathbb{B}_{3/4}(a)} u_r) \leq \frac{C}{r^{\frac{2(p+1)}{pq-1}}}. \tag{4.13}$$

If $pq = 1$, A contradiction is obviously arrived. We readily infer that $u \equiv v \equiv 0$.

While $pq > 1$, we observe that

$$\inf_{\mathbb{B}_{3/4}(a)} v_r = \inf_{\mathbb{B}_{3r/4}(ar)} v \geq \frac{r}{4} \inf_{\mathbb{B}_{3r/4}(ar)} \frac{v}{x_n} \geq \frac{r}{4} \inf_{\mathbb{B}_{2r}} \frac{v}{x_n} = \frac{r}{4} m_v(2r), \tag{4.14}$$

$$\inf_{\mathbb{B}_{3/4}(a)} u_r = \inf_{\mathbb{B}_{3r/4}(ar)} u \geq \frac{r}{4} \inf_{\mathbb{B}_{3r/4}(ar)} \frac{u}{x_n} \geq \frac{r}{4} \inf_{\mathbb{B}_{2r}} \frac{u}{x_n} = \frac{r}{4} m_u(2r). \tag{4.15}$$

From (4.12) and (4.14), we obtain

$$r^{\tilde{n}} m_v(r) \leq \frac{C}{r^{\frac{2(q+1)}{pq-1}+1-\tilde{n}}}. \tag{4.16}$$

By (4.13) and (4.15), we have

$$r^{\tilde{n}} m_u(r) \leq \frac{C}{r^{\frac{2(p+1)}{pq-1}+1-\tilde{n}}}. \tag{4.17}$$

If

$$\frac{2(p+1)}{pq-1} > \tilde{n} - 1 \;\text{ or }\; \frac{2(q+1)}{pq-1} > \tilde{n} - 1,$$

then $r^{\tilde{n}} m_u(r) \to 0$ or $r^{\tilde{n}} m_v(r) \to 0$ as $r \to \infty$. Hence Lemma 4.2 shows that $u \equiv 0$ or $v \equiv 0$. From the structure of fully nonlinear elliptic equation systems, we obtain $u \equiv 0$ and $v \equiv 0$ in either of the cases.

Next we study the critical case that

$$\frac{2(p+1)}{pq-1} = \tilde{n} - 1 \ \text{ and } \ \frac{2(q+1)}{pq-1} = \tilde{n} - 1.$$

It is easy to check that $p = q = \frac{\tilde{n}+1}{\tilde{n}-1}$. In this case, (4.16) and (4.17) become

$$r^{\tilde{n}} m_v(r) \leq C \qquad \forall r > 0 \tag{4.18}$$

and

$$r^{\tilde{n}} m_u(r) \leq C \qquad \forall r > 0. \tag{4.19}$$

Thanks to the monotonicity property of $r^{\tilde{n}} m_u(r)$ in Lemma 4.2,

$$r^{\tilde{n}} m_u(r) \geq r_0^{\tilde{n}} m_u(r_0) \quad \text{ for } r \geq r_0.$$

Then

$$u(x) \geq C \frac{x_n}{r^{\tilde{n}}} \quad \text{ for } x \in \mathbb{R}^n_+ \backslash \mathbb{B}_{r_0}. \tag{4.20}$$

With the aid of (4.20),

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) \geq C\left(\frac{x_n}{r^{\tilde{n}}}\right)^{\frac{\tilde{n}+1}{\tilde{n}-1}}, \quad \forall x \in \mathbb{R}^n_+ \backslash \mathbb{B}_{r_0}. \tag{4.21}$$

Taking into account of Lemma 4.3,

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2(\gamma\Gamma)) \leq -\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) \tag{4.22}$$

is satisfied by appropriately chosen γ. Choosing

$$\gamma \leq m_v(r_0) \frac{r_0^{\tilde{n}_1}}{elnr_0 + f},$$

we have

$$\gamma\Gamma(x) \leq v(x) \quad \text{ on } \partial\mathbb{B}_{r_0}.$$

For any fixed $\epsilon > 0$, let $R > 0$ be so large that

$$\gamma\Gamma(x) \leq \epsilon \quad \text{ for } \mathbb{R}^n_+ \backslash \mathbb{B}_R.$$

The comparison principle in Lemma 2.1 for $\gamma\Gamma(x)$ and $v(x) + \epsilon$ in $\mathbb{B}_R \backslash \mathbb{B}_{r_0}$ shows that

$$\gamma\Gamma(x) \leq v(x) + \epsilon.$$

In addition, let $R \to \infty$ and then $\epsilon \to 0$, we have

$$\gamma\Gamma(x) \leq v(x) \quad \forall x \in \mathbb{R}^n_+ \backslash \mathbb{B}_{r_0}.$$

From the explicit form of $\Gamma(x)$,

$$v(x) \geq C \frac{x_n}{|x|^{\tilde{n}}} ln|x| \quad \forall x \in \mathbb{R}^n_+ \backslash \mathbb{B}_{r_0},$$

which implies that

$$m_v(r)r^{\tilde{n}} \geq C\ln r \quad \forall r \geq r_0.$$

It contradicts the bound in (4.18). The theorem is thus accomplished.

The proof of Corollary 1.2 is the consequence of the above arguments and estimates in [14]. We omit it here.

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